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Original Research Article

Development of Rayleigh-Exponentiated Odd Generalized-Weibull Distribution with Properties

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ABSTRACT

We propose a new family of distributions called the Rayleigh-Exponentiated odd Generalized-Weibull Distribution with two positive parameters, which generalizes the Cordeiro and de Castro's family. Some special distributions in the new class are discussed. We derive the validity and some mathematical properties of the proposed distribution including explicit expressions for the quantile function, ordinary moments, Moment generating function, hazard and Survival function. The method of maximum likelihood is used to estimation of REOG-Weibull distributions. These functions are illustrated with graphs, since REOGWD was evident through graphics.

INTRODUCTION

In recent years, statistical distributions have become fundamental tools in modelling complex real- world phenomena across various fields, including survival analysis, reliability engineering, and financial risk management (Johnson et al., 1994). Classical distributions such as the Weibull and Rayleigh distributions have been widely used in these areas due to their simplicity and interpretability (Lawless, 2003). However, these conventional models sometimes fail to capture the heavy tails and skewness found in practical data, limiting their effectiveness (Mudholkar et al., 1996).

To address these limitations, several generalized families of distributions have been proposed to improve the flexibility of these classical models. Among them, the Exponentiated-Weibull and Odd Generalized-Weibull distributions provide additional shape parameters, allowing greater adaptability to various data patterns (Gupta & Kundu, 1999; Al-Babtain et al., 2020). These flexible distributions have proven effective in fitting skewed and kurtotic data, which is common in survival and reliability studies, but they still sometimes lack flexibility in capturing diverse tail behaviours (Louzada-Neto et al., 2001).

The Rayleigh distribution, initially developed for modelling energy dissipation in physical systems, has shown versatility in various applications but lacks the flexibility of higher-order distributions (Rayleigh, 1880). Extending this distribution through exponentiation and further generalization can enhance its suitability for broader applications, especially where data exhibit heavy tails (Akinsete et al., 2008). The development of the Rayleigh-Exponentiated Odd Generalized-Weibull (REOGW) distribution is, therefore, a promising extension. This distribution combines the properties of the Rayleigh and Odd Generalized-Weibull families, aiming to offer increased adaptability for survival analysis and real-life datasets.

The primary objective of this study is to develop the PDF and CDF of the REOGW distribution, addressing the need for more adaptable models in fields that require precise modelling of survival times and reliability (Pham, 2006). This involves not only deriving the core functions but also establishing essential mathematical properties, such as moments, which can reveal critical distributional features relevant to data patterns observed in survival studies (Cohen & Whitten, 1988).

Parameter estimation through maximum likelihood estimation (MLE) will further solidify the practical efficacy of this model, as MLE is widely regarded as a reliable method for parameter estimation in complex distributional structures (Azzalini, 1985). MLE’s consistency and asymptotic properties align well with the model’s goal of accurately capturing the distribution’s behaviour over different datasets (Bishop, 2006).

The study’s comparative analysis with existing models will help determine the REOGW distribution’s relative performance across diverse datasets. This step is essential for validating its effectiveness and understanding its practical limitations and strengths (Mendenhall et al., 2016). Conducting simulations for parameter consistency will also be critical, as simulation studies are widely used to assess the robustness of new distributions in the statistical literature (Lee, 2010).

On Theoretical Study of Rayleigh-Exponentiated Odd Generalized-X Family of Distributions, Yahaya and Doguwa (2021) developed the Rayleigh-Exponentiated Odd Generalized (REOG-X) family by deriving new theoretical properties and establishing a versatile framework applicable across multiple distributional forms. This study delves into the mathematical development of the REOG-X family, providing detailed derivations of its probability density function (PDF), cumulative distribution function (CDF), moments, and hazard functions. The authors showcase the REOG-X family’s ability to model diverse data patterns, including skewed and heavy-tailed distributions, and validate its adaptability through practical applications. By using the method of maximum likelihood estimation (MLE) for parameter estimation, the study affirms the REOG-X family's capacity to fit data with greater precision compared to conventional models. This work advances statistical modelling by offering a highly flexible family of distributions that can accurately capture complex real-world phenomena, with significant implications for fields like survival analysis and reliability engineering. This research is a noteworthy contribution, providing a robust statistical tool for modelling datasets with intricate structures and extreme events.

The study by (Yahaya and Doguwa, 2022), titled On Rayleigh-Exponentiated Odd Generalized- Pareto Distribution with its Applications, presents a novel statistical distribution aimed at enhancing flexibility in modelling complex data patterns, particularly those exhibiting skewness, heavy tails, and varying hazard rates. The authors introduce the Rayleigh-Exponentiated Odd Generalized-Pareto (REOGP) distribution, extending traditional models to better capture a wider range of real-world behaviours. Key mathematical properties, including the probability density function (PDF), cumulative distribution function (CDF), moments, and hazard functions, are rigorously derived, demonstrating the distribution's robustness and adaptability. Through parameter estimation via the maximum likelihood method, the study confirms the distribution’s ability to fit data more effectively than several traditional distributions, especially in fields such as survival analysis and reliability engineering. Their simulation studies provide evidence of the consistency and accuracy of the parameter estimates, further solidifying the REOGP’s applicability. The study contributes to statistical literature by proposing a highly adaptable distribution that can address the limitations of existing models, with broad implications for data analysis in practical applications requiring precise modelling of extreme events.

In the study “On the Properties and Applications of a New Extension of Exponentiated Rayleigh Distribution”, (Abdulsalam et al., 2021) introduce an innovative extension of the Exponentiated Rayleigh (ER) distribution, designed to improve flexibility in modelling diverse data types. The authors rigorously derive key properties, including the probability density function (PDF), cumulative distribution function (CDF), moments, and reliability measures, illustrating the distribution’s enhanced capacity to capture skewness and heavy tails compared to traditional models.

The study employs maximum likelihood estimation (MLE) for parameter estimation, establishing the consistency and robustness of the model in various applications. Through simulation and real-world data applications, the extended distribution outperforms standard models in fitting complex datasets, especially in reliability analysis and survival studies. This research contributes a powerful, flexible tool to the statistical literature, expanding the applicability of the Exponentiated Rayleigh family in fields requiring accurate modelling of non-standard data behaviours.

The study “Applications of Inverse Weibull Rayleigh Distribution to Failure Rates and Vinyl Chloride Data Sets” by Adamu et al. (2021) explores the Inverse Weibull Rayleigh (IWR) distribution’s application in modelling failure rates and chemical exposure data. The authors present a thorough analysis of the IWR distribution, deriving essential properties, including the probability density function (PDF), cumulative distribution function (CDF), and hazard function, to demonstrate its suitability for modelling heavy-tailed data with complex failure rates. Through maximum likelihood estimation (MLE) for parameter fitting, the study assesses the IWR distribution's performance on vinyl chloride exposure and failure rate datasets, showing improved accuracy over traditional models. This research contributes meaningfully to reliability and environmental studies by offering a robust alternative for analyzing extreme events and risk behaviours, broadening the practical applications of the Inverse Weibull Rayleigh model in real- world datasets.

In “Generalized Rayleigh Distribution: Different Methods of Estimations”, Kundu and Raqab (2005) provide an in-depth exploration of estimation techniques for the Generalized Rayleigh Distribution (GRD), a model known for its flexibility in representing reliability and life data. The authors systematically compare estimation methods, including maximum likelihood estimation (MLE), method of moments, and Bayes estimators, assessing each method’s efficiency, bias, and mean square error through simulation studies. Their findings reveal MLE as the most robust technique for GRD parameter estimation, especially for large samples, while Bayes estimators offer advantages with smaller datasets or prior information. This work is significant in statistical modelling, presenting the GRD as a powerful alternative to traditional models for data with non- standard shapes, such as skewed or heavy-tailed distributions. By evaluating estimation accuracy across methods, Kundu and Raqab provide a practical guide for applying the GRD in real-world scenarios, particularly in reliability engineering and survival analysis

In “Rayleigh Distribution and Its Generalizations”, Beckmann (1964) explores the foundational Rayleigh distribution and introduces generalized forms to extend its application across various scientific domains. Beckmann presents mathematical properties and distributional behaviour, emphasizing the versatility of the Rayleigh model in representing wave intensities and scattered signal amplitudes, particularly in radio science. By generalizing the Rayleigh distribution, Beckmann opens avenues for more accurate modelling of complex, real-world phenomena, such as environmental noise and signal processing in communications. This early work is pivotal, as it establishes the Rayleigh distribution’s adaptability, setting a foundation for later developments in statistical distributions used in reliability engineering, survival analysis, and environmental science. Beckmann’s study remains influential, highlighting the Rayleigh distribution’s robustness and the potential of its generalizations to improve data modelling accuracy across scientific fields.

In Two-Parameter Rayleigh Distribution: Different Methods of Estimation, Dey et al., (2014) examine various estimation techniques for the two-parameter Rayleigh distribution, which is widely used in modelling reliability and life data. The authors compare methods including maximum likelihood estimation (MLE), method of moments, and percentile-based estimations, assessing each for efficiency, bias, and mean square error through extensive simulation studies. Their results indicate MLE as the most effective method for parameter estimation in larger samples, while percentile-based methods offer a viable alternative for small datasets. This study makes a valuable contribution to statistical modelling by detailing practical approaches to estimating parameters in the two-parameter Rayleigh model, which has applications in fields like engineering and environmental sciences. By offering a comprehensive comparison of estimation methods, the authors provide essential insights for researchers and practitioners needing precise data modelling with the Rayleigh distribution.

In The Theory of the Rayleigh Distribution and Some of Its Applications”, Hoffman and Karst (1975) provide a thorough exploration of the Rayleigh distribution, highlighting its theoretical basis and practical applications, particularly in marine and ship research. The authors detail the distribution’s mathematical properties, including its probability density function, moments, and estimation techniques, underscoring its relevance in modelling wave heights, signal processing, and load distributions in marine environments. Their application-focused approach demonstrates the Rayleigh distribution’s utility in describing stochastic phenomena typical in ship and ocean engineering. This study is influential, establishing the Rayleigh distribution as a robust model for analyzing random and complex data patterns encountered in maritime contexts. Hoffman and Karst's work remains a key reference, showcasing the distribution’s versatility and its critical role in advancing research and applications in marine science and engineering.

In An Extension of Rayleigh Distribution and Applications, Ateeq et al., (2019) introduce an extended Rayleigh distribution designed to enhance modelling flexibility for datasets with diverse characteristics, including skewed and heavy-tailed distributions. The authors derive critical mathematical properties, such as the probability density function, cumulative distribution function, and moments, and validate the distribution’s applicability through real-world datasets in fields like reliability engineering and survival analysis. Parameter estimation is carried out using maximum likelihood estimation (MLE), demonstrating the extended model’s effectiveness in fitting complex data. This study contributes significantly to statistical modelling by offering a robust alternative to the traditional Rayleigh distribution, with improved adaptability for non-standard data patterns. The findings emphasize the extension’s practical value across disciplines that require precise modelling of variability, making it a useful tool in applied statistics.

MATERIALS AND METHODS

Here, we present the methodology for extending the Weibull distribution using the REOG-X family of distribution developed by Yahaya and Doguwa (2021). The extended Weibull model is called the REOG-Weibull distribution. The statistical properties related to this new distribution are derived and presented. The method of estimating the parameters of this family is presented. Usman et al. (2021) defined a random variable T and is said to have Weibull distribution with shape parameter β and scale parameter λ if its CDF and pdf are given as,

$$G(t;\lambda,\beta) = 1 - exp\{-\lambda t^{\beta}\}; t,\beta,\lambda < 0 \quad (1)$$

$$g(t;\lambda,\beta) = \beta\lambda t^{\beta-1}\, exp\{-\lambda t^{\beta}\}; t,\beta,\lambda < 0 \quad (2)$$

Yahaya and Doguwa (2021) defined a random variable T which is said to have a Rayleigh Exponentiated Odd Generalized-T family of distribution with two shape parameters θ and α if its pdf and CDF are given as,

$$F(t;\alpha,\theta) = 1 - exp\left\{-\frac{\theta}{2}\left(\frac{G^{\alpha}(t;\xi)}{(1-G^{\alpha}(t;\xi))}\right)^{2}\right\}; t,\theta,\alpha > 0 \quad (3)$$

$$f(t;\alpha,\theta) = \frac{\alpha\theta g(t;\xi)G^{2\alpha-1}(t;\xi)}{(1-G^{\alpha}(t;\xi))^{3}} exp\left\{-\frac{\theta}{2}\left(\frac{G^{\alpha}(t;\xi)}{(1-G^{\alpha}(t;\xi))}\right)^{2}\right\}; t,\theta,\alpha > 0 \quad (4)$$

The Proposed REOG-Weibull Distribution

We developed a new distribution called the Rayleigh Exponentiated Odd Generalized-Weibull (REOG-W) distribution by substituting equation (2) into equation (4) to obtain the CDF given by;

$$F(t;\alpha,\beta,\lambda,\theta) = 1 - exp\left\{-\frac{\theta}{2}\left(\left(1 - exp\{-\lambda t^{\beta}\}\right)^{-\alpha} - 1\right)^{-2}\right\}; t,\theta,\alpha > 0 \quad (5)$$

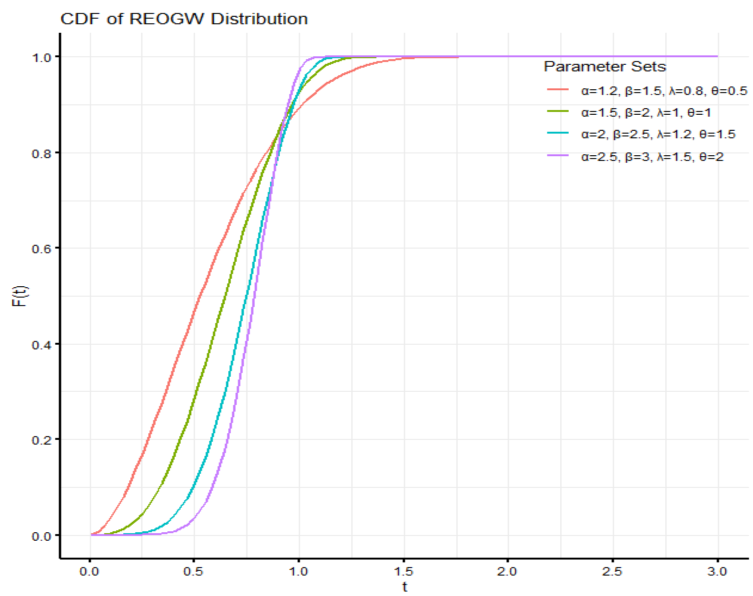

Figure 1: CDF Plot of REOG-Weibull Distribution

The corresponding pdf of equation (5) is obtained by differentiating the CDF with respect to t as;

$$f(t;\alpha,\beta,\lambda,\theta) = \frac{\alpha\beta\lambda\theta t^{\beta-1} exp\{-\lambda t^{\beta}\}(1-exp\{-\lambda t^{\beta}\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^{\beta}\})^{\alpha}\right)^{3}} exp\left\{-\frac{\theta}{2}\left(\left(1 - exp\{-\lambda t^{\beta}\}\right)^{-\alpha} - 1\right)^{-2}\right\} \quad (6)$$

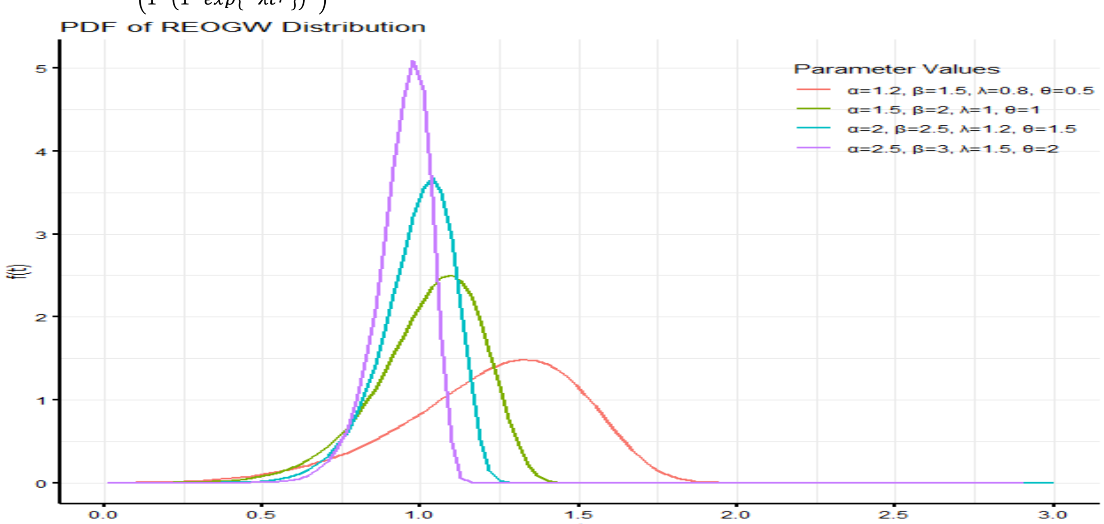

Figure 2: PDF Plot of REOG-Weibull Distribution

Validity Test of the PDF

$\int_{-\infty}^{\infty} f(t)dt = 1$

$f(t;\alpha,\beta,\lambda,\theta)dt = 1$

From the LHS, we have;

$$= \int_{-\infty}^{\infty} \frac{\alpha\beta\lambda\theta t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\} dt \quad (7)$$

From equation 7

Let $\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}$

Using the chain rule, we have;

$\frac{dy}{dm} = -2m^{-3} \Rightarrow m = \left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)$

$\frac{dm}{dt} = -\alpha\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha-1}\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}$

$\frac{dy}{dt} = \frac{dy}{dm} \times \frac{dm}{dt}$

$= -2m^{-3}\left(-\alpha\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha-1}\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\right)$

$= 2\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha-1} m^{-3}$

But, $m = \left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)$

$$\frac{dy}{dt} = \frac{2\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha-1}}{\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{3}}$$

But, $\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{3} = \frac{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}}{\left(1-exp\{-\lambda t^{\beta}\}\right)^{3\alpha}}$

$$\frac{dy}{dt} = \frac{2\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha-1}\left(1-exp\{-\lambda t^{\beta}\}\right)^{3\alpha}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}}$$

$$= \frac{2\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}}$$

$$dt = \frac{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3} dy}{2\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}} \quad (8)$$

Substituting equation (8) into equation (7) we have;

$\frac{\theta}{2}\int_{0}^{\infty} exp\left\{-\frac{\theta}{2}y\right\} dy$

Let $z = \frac{\theta}{2}y \Rightarrow \frac{dz}{dy} = \frac{\theta}{2} \Rightarrow dy = \frac{2dz}{\theta}$

$= \frac{\theta}{2}\int_{0}^{\infty} exp\{-z\}\frac{2dz}{\theta}$

$= \frac{\theta}{2}\int_{0}^{\infty} exp\{-z\} = -[exp\{-z\}]_{0}^{\infty} = -[exp\{-\infty\} - exp\{0\}]_{0}^{\infty} = -[0-1] = 1$

Validity Test of the CDF

For $t = 0$ we have

$\lim_{t\to 0} F(t) = 0$

$\lim_{t\to 0} F(t;\alpha,\beta,\lambda,\theta) = 0$

From equation (3.8)

$\lim_{t\to 0}\left(1-exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}\right) = 0$

For $t = \infty$ we have

$\lim_{t\to\infty} F(t) = 1$

$\lim_{t\to\infty} F(t;\alpha,\beta,\lambda,\theta) = 1$

From equation (3.5)

$\lim_{t\to\infty}\left(1-exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}\right) = 1$

Survival and Hazard Rate Function of the OR-G Family

The survival function, and hazard function of a random variable T which follows the REOG-Weibull distributions are respectively given as,

$S(t;\alpha,\beta,\lambda,\theta) = 1 - F(t;\alpha,\beta,\lambda,\theta)$

$= 1 - \left(1-exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}\right)$

$$= exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\} \quad (9)$$

The hazard function is given as;

$h(t;\alpha,\beta,\lambda,\theta) = \frac{f(t;\alpha,\beta,\lambda,\theta)}{S(t;\alpha,\beta,\lambda,\theta)}$

$$= \frac{\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}$$

$$= \frac{\alpha\beta\lambda t^{\beta-1} exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}} \quad (10)$$

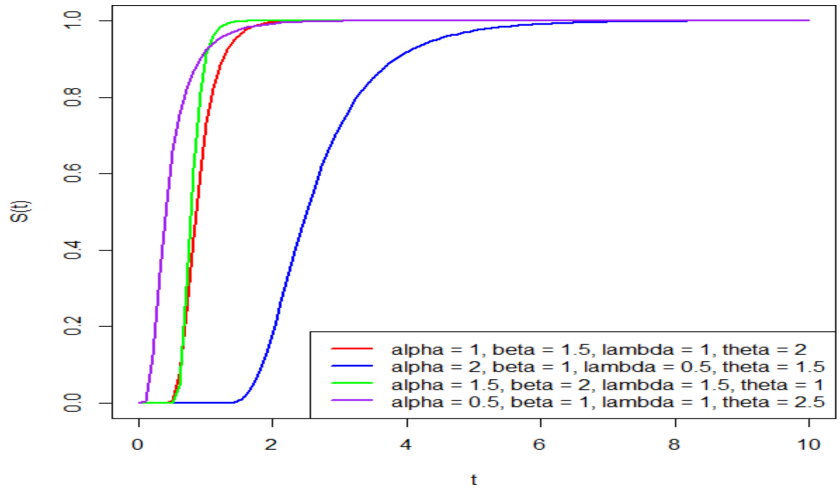

Figure 3: Survival Function Plot of REOG-Weibull Distribution

Quantile Function of REOG-Weibull Distribution

To derive the quantile function of the Rayleigh-Exponentiated Odd Generalized Weibull (REOGW) distribution from its cumulative distribution function (CDF), we follow these steps:

Step 1: Start with the CDF

Given the CDF of the REOGW distribution in equation (5):

$$F(t;\alpha,\beta,\lambda,\theta) = 1 - exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}$$

Step 2: Set the CDF Equal to a Probability

To find the quantile function $Q(p)$, set the CDF equal to p where $0 < p < 1$:

$$p = 1 - exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}$$

Steps 3: Solve for t

Rearrange to isolate the exponential term:

$$exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\} = p-1$$

Take the natural logarithm of both sides:

$$-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2} = In(p-1)$$

Simplify to isolate the expression involving t

$$\left[\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right]^{-2} = \frac{2In(p-1)}{\theta}$$

Invert the power of -2 to simplify

$$\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1 = \sqrt{\frac{2In(1-p)}{\theta}}$$

Solve for $\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}$:

$$\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha} = 1+\sqrt{-\frac{2In(1-p)}{\theta}}$$

Invert to isolate $\left[exp\{-\lambda t^{\beta}\}\right]$:

$$1-exp\{-\lambda t^{\beta}\} = \left(1+\sqrt{-\frac{2In(1-p)}{\theta}}\right)^{-\frac{1}{\alpha}}$$

Solve for $exp\{-\lambda t^{\beta}\}$:

$$exp\{-\lambda t^{\beta}\} = 1-\left(1+\sqrt{-\frac{2In(1-p)}{\theta}}\right)^{-\frac{1}{\alpha}}$$

Take the natural logarithm:

$$-\lambda t^{\beta} = In\left[1-\left(1+\sqrt{-\frac{2In(1-p)}{\theta}}\right)^{-\frac{1}{\alpha}}\right]$$

Solve for t:

$$t = \left(-\frac{1}{\lambda}In\left[1-\left(1+\sqrt{-\frac{2In(1-p)}{\theta}}\right)^{-\frac{1}{\alpha}}\right]\right)^{\frac{1}{\beta}}$$

Finally, the Quantile Function is given as:

$$Q(p) = \left(-\frac{1}{\lambda}In\left[1-\left(1+\sqrt{-\frac{2In(1-p)}{\theta}}\right)^{-\frac{1}{\alpha}}\right]\right)^{\frac{1}{\beta}} \quad (11)$$

Equation (3.11) is the quantile function $Q(p)$of the REOGW distribution.

Moments of REOG-Weibull Distribution

To derive the moments of the Rayleigh-Exponentiated Odd Generalized Weibull (REOGW) distribution, we need to evaluate the rth moment $E(T^r)$using the given probability density function (PDF) in equation (6). The rth moment is defined by:

$$E(T^r) = \int_0^{\infty} t^r f(t;\alpha,\beta,\lambda,\theta)dt \quad (12)$$

Where $f(t;\alpha,\beta,\lambda,\theta)$ is the PDF of the REOGW distribution given in equation (6):

Steps we follow to derive the Moments:

Substitute the PDF into the Moment Definition in equation (12):

$$E(T^r) = \int_0^\infty t^r \frac{\alpha\beta\lambda\theta t^{\beta-1} exp\{-\lambda t^\beta\}(1-exp\{-\lambda t^\beta\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^\beta\})^\alpha\right)^3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\} dt$$

Simplify the integral:

$$E(T^r) = \alpha\beta\lambda\theta \int_0^\infty t^{r+\beta-1} \frac{exp\{-\lambda t^\beta\}(1-exp\{-\lambda t^\beta\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^\beta\})^\alpha\right)^3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\} dt$$

Transformation for Simplification:
Let $u = exp\{-\lambda t^\beta\} \Rightarrow du = -\lambda t^{\beta-1}\, exp\{-\lambda t^\beta\}\, dt$
Solve for dt:

$$dt = -\frac{du}{\lambda t^{\beta-1} u}$$

Substitute and re-express the Moment integral $t^\beta = -\frac{Inu}{\lambda} and dt^{\beta-1} = \left(-\frac{Inu}{\lambda}\right)^{\frac{\beta-1}{\beta}}$

$$E(T^r) = \alpha\beta\lambda\theta \int_0^1 \left(-\frac{Inu}{\lambda}\right)^{\frac{r+\beta-1}{\beta}} \frac{u(1-u)^{2\alpha-1}}{(1-(1-u)^\alpha)^3} exp\left\{-\frac{\theta}{2}((1-u)^{-\alpha}-1)^{-2}\right\}\left(-\frac{du}{\lambda\beta\left(-\frac{Inu}{\lambda}\right)^{\frac{\beta-1}{\beta}} u}\right)$$

We simplify the above expression by cancelling and rearranging terms:

$$E(T^r) = \alpha\theta \int_0^1 \left(-\frac{Inu}{\lambda}\right)^{\frac{r}{\beta}} \frac{u(1-u)^{2\alpha-1}}{(1-(1-u)^\alpha)^3} exp\left\{-\frac{\theta}{2}((1-u)^{-\alpha}-1)^{-2}\right\}\frac{du}{u} \tag{13}$$

Moment-generating Function of REOG-Weibull Distribution

To derive the moment-generating function (MGF) $M_T(s)$ of the REOGW distribution, we start from the definition:

$$M_T(s) = E(e^{st}) = \int_0^\infty e^{st} f(t;\alpha,\beta,\lambda,\theta)dt \tag{14}$$

Where $f(t;\alpha,\beta,\lambda,\theta)$ is the PDF of the REOGW distribution given in equation (6):
We follow step-by-step Derivation as:
Substitute the PDF into the MGF definition:

$$M_T(s) = \int_0^\infty e^{st} \frac{\alpha\beta\lambda\theta t^{\beta-1} exp\{-\lambda t^\beta\}(1-exp\{-\lambda t^\beta\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^\beta\})^\alpha\right)^3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\} dt$$

Simplify the exponential term e^{st}
The term e^{st} remains as in the integrand. The integral now becomes:

$$M_T(s) = \alpha\beta\lambda\theta \int_0^\infty t^{\beta-1} e^{st}\, exp\{-\lambda t^\beta\} \frac{(1-exp\{-\lambda t^\beta\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^\beta\})^\alpha\right)^3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\} dt \tag{15}$$

Order Statistics of REOG-Weibull Distribution

We derive the distribution of the order statistics for the REOGW (Rayleigh-Exponentiated Odd Generalized Weibull) distribution, we need to find the PDF of the k^{th} order statistic from a sampleof size n.The distribution's PDF and CDF are provided as follows:
The PDF of the kth order statistic $X_{(k)}$ in a sample of size n is given by:

$$f_{X(k)}(t) = \frac{n!}{(k-1)!(n-k)!}[F(t)]^{k-1}[1-F(t)]^{n-k} f(t), \tag{16}$$

Substitute the CDF and PDF of the REOGW distribution in equation (5) and (6) into the order
Order statistic formula in equation (16) we have;

$$f_{X(k)}(t) = \frac{n!}{(k-1)!(n-k)!}\left[1-exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\}\right]^{k-1}\left[exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\}\right]^{n-k}$$
$$\frac{\alpha\beta\lambda\theta t^{\beta-1} exp\{-\lambda t^\beta\}(1-exp\{-\lambda t^\beta\})^{2\alpha-1}}{\left(1-(1-exp\{-\lambda t^\beta\})^\alpha\right)^3} exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^\beta\}\right)^{-\alpha}-1\right)^{-2}\right\} \tag{17}$$

MLE Estimation of REOG-Weibull Distribution

To derive the log-likelihood function of the Rayleigh-Exponentiated Odd Generalized Weibull(REOGW) distribution, we start from the given PDF in equation (6). Given a sample of size $n, \{t_1, t_2, \ldots\ldots t_n\}$, the log-likelihood function $\ell(\alpha,\beta,\lambda,\theta)$ is defined as:

$$\ell(\alpha,\beta,\lambda,\theta) = \sum_{i=1}^{n\sum(t;\alpha,\beta,\lambda,\theta)} log \tag{18}$$

Substituting the PDF in the equation (6) into the log-likelihood function, we have;

$$\ell(\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n\Sigma}\left[\frac{\alpha\beta\lambda\theta t^{\beta-1}\,exp\{-\lambda t^{\beta}\}\left(1-exp\{-\lambda t^{\beta}\}\right)^{2\alpha-1}}{\left(1-\left(1-exp\{-\lambda t^{\beta}\}\right)^{\alpha}\right)^{3}}exp\left\{-\frac{\theta}{2}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2}\right\}\right]log$$

Separating the terms inside the logarithm we have;

$$\ell(\alpha,\beta,\lambda,\theta)=\left[\sum_{i=1}^{n}log(\alpha)+log(\beta)+log(\lambda)+log(\theta)+(\beta-1)\,log(t_i)-\lambda t_i{}^{\beta}+(2\alpha-1)-3\log(1-(1-\exp\{-\lambda t_i{}^{\beta}\})^{\alpha})-\frac{\theta}{2}((1-\exp\{-\lambda t^{\beta}\})^{-\alpha}-1)^{-2}\right]$$

The log-likelihood function can be written as:

$$(\alpha,\beta,\lambda,\theta)=n\,log(\alpha)+log(\beta)+log(\lambda)+log(\theta)+(\beta-1)\sum_{i=1}^{n}log(t_i)-\lambda\sum_{i=1}^{n}t_i+(2\alpha-1)\sum_{i=1}^{n\Sigma(1-exp\{-\lambda t_i{}^{\beta}\})}log-3\sum_{i=1}^{n}log\left(1-\left(1-exp\{-\lambda t_i{}^{\beta}\}\right)^{\alpha}\right)-\frac{\theta}{2}\sum_{i=1}^{n}\left(\left(1-exp\{-\lambda t^{\beta}\}\right)^{-\alpha}-1\right)^{-2} \quad (19)$$

This equation (19) can be used for further analysis or numerical estimation of the parameter $\alpha,\beta,\lambda,\theta$.

RESULTS AND DISCUSSION

Monte Carlo Simulation

We present the results and discussion of the Monte Carlo Simulations for the REOG-Weibull model, applications to real-life datasets for the model and competing models. Given the three competing models, the Weibull model, the exponential Weibull model, and the power Rayleigh model we can carry out the model fitting and comparison using the real-life datasets.

The Monte Carlo simulation results in Table 1 provide a comprehensive evaluation of the estimation process for the REOG-Weibull distribution across different sample sizes. The results show that as the sample size increases, the estimates for parameters $\alpha,\beta,\lambda,$ and θ approach the true values more closely, which is reflected in the decreasing bias and RMSE values.

Table 1: Results of the simulated data from the REOG-Weibull Distribution

Sample Size	Parameters	Estimates	|Bias|	RMSE
20	α	0.2238	0.2762	0.2793
	β	0.2674	0.2250	0.2353
	λ	0.2740	0.2003	0.2075
	θ	0.3247	0.2476	0.2541
50	α	0.2314	0.2686	0.2717
	β	0.2808	0.2192	0.2250
	λ	0.2743	0.1888	0.1988
	θ	0.3373	0.2464	0.2526
100	α	0.2338	0.2662	0.2691
	β	0.2860	0.2140	0.2212
	λ	0.2796	0.1843	0.1963
	θ	0.3429	0.2462	0.2521
200	α	0.2410	0.2590	0.2616
	β	0.3051	0.1949	0.2060
	λ	0.2843	0.1796	0.1950
	θ	0.3462	0.2429	0.2510
250	α	0.2432	0.2568	0.2590
	β	0.3243	0.1757	0.1933
	λ	0.2888	0.1743	0.1869
	θ	0.3464	0.2373	0.2469
500	α	0.2498	0.2502	0.2516
	β	0.3793	0.1207	0.1666
	λ	0.3003	0.1740	0.1859
	θ	0.3476	0.2247	0.2363

This pattern indicates that the estimators for these parameters are consistent, as they converge to the true parameter values with increasing sample size and lower RMSE and bias at larger sample sizes show that the estimates are not only close to the true values but also exhibit less dispersion.

Application to Real-Life Datasets

We used some of the existing datasets to compare the performance of the developed distribution and other related distributions. To fit our developed distribution and comparator distributions, the competing models are: the Weibull-Weibull (WW) distribution developed by Bourguinon *et al*. (2014); the New Weibull-Weibull (NWW) distribution developed by Tahir *et al*. (2016); the Generalized Odd Frechet-Weibull (GOFW) distribution developed by Margapool *et al*. (2020).

Table 2: Goodness of Fit Measures for AAML Dataset

Model	Log-Likelihood	AIC	P-value
REOGW	-90.27824	188.5565	2.78e-80
WW	-98.36816	204.7363	3.00e-65
NWW	-95.65894	199.3179	3.84e-68
GOFW	-390.26233	786.5247	4.54e-73

Table 2 presents the goodness-of-fit measures for four different survival regression models applied to the AAML dataset. The models are evaluated using Log-Likelihood, Akaike Information Criterion (AIC), and p-values from a goodness-of-fit test.

Based on the log-likelihood, AIC, and p-value: The REOGW model is the best fit for the AAML dataset. The GOFW model is the worst fit, performing significantly worse than the others. Between WW and NWW, NWW fits slightly better than WW due to its higher log-likelihood and lower AIC.

CONCLUSION

This study introduced the Rayleigh-Exponentiated Odd Generalized Weibull (REOGW) distribution as a new model for survival data analysis. We explored its fundamental statistical properties and employed the Maximum Likelihood Estimation (MLE) method for parameter estimation. Through extensive simulation studies, we demonstrated the efficiency and consistency of the estimators.

The performance of the REOGW distribution was assessed using real-life survival datasets, including remission times of bladder cancer patients and survival times of patients with Advanced Acute Myelogenous Leukemia (AAML-1). The goodness-of-fit measures, including Log-Likelihood, Akaike Information Criterion (AIC), and P-values, showed that the REOGW model outperformed the Weibull-Weibull (WW), New Weibull-Weibull (NWW), and Generalized Odd Frechet Weibull (GOFW) distributions. These results confirmed that the REOGW distribution provides a superior fit for survival data, demonstrating flexibility in capturing different hazard rate structures. Overall, the REOGW model presents a significant advancement in survival analysis, offering a more robust and accurate tool for modelling survival data in medical research and reliability studies. Future studies may explore extensions of the REOGW distribution and its applications in broader fields, such as engineering reliability and financial risk analysis.

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