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Original Research Article

A Spectral Conjugate Gradient Method via Hybridization Approach for System of Nonlinear Equations

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ABSTRACT
This paper presents an effective conjugate gradient method via hybridization approach of classical Newton direction and conjugate gradient search direction, the method scheme satisfies the sufficient decent condition. Under mild condition, the global convergence result for the method is established. Preliminary numerical results for some large-scale benchmark test problems reported in this work, demonstrate that, the method is practically effective and competitive to some existing methods.

INTRODUCTION

Consider the general form of system of nonlinear equations:

$$F(x) = 0 \tag{1}$$

Where $F: \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map which assumed to be continuously differentiable functions. The system of nonlinear equations arises in many areas of scientific computing and engineering applications. A variety of different iterative methods have been developed for solving problem (1), for example, Newton's method, quasi-Newton method, Gauss-Newton Method Abubakar, (2018) and their variants. However, they are not particularly suitable for solving large-scale problems, because they need to solve linear system of equations using Jacobian matrix or its approximation at each iteration. It is vital to mention that, the conjugate gradient (CG) methods are among the popular methods used to solve large-scale system of nonlinear equations, due to their rapid convergence property, simple to implement and low storage requirement Yu-Hong, (1999) and Zhen-Jun, (2008) In fact, conjugate gradient method has played a vital role in solving optimization problems.

However, it generates a sequence of iterative points $\{x_k\}$ from an initial guess $x_0 \in \mathbb{R}^n$, using the iterative formula in Can, (2013)

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0,1,2, \dots \tag{2}$$

Where x_k is the previous iterative point, x_{k+1} is the current iterative point, $\alpha_k > 0$ is the step-length computed via any suitable line search technique and d_k is the search direction defined by:

$$d_k = \begin{cases} -F(x_k), & if \;\; k = 0, \\ -F(x_k) + \beta_k d_{k-1}, & if \;\; k \geq 1 \end{cases} \tag{3}$$

Where $F_k = F(x_k)$ and β_k is termed as conjugate gradient parameter. Conjugate gradient methods differ in their way of defining the CG update parameter β_k because different choices of β_k give rise to distinct conjugate gradient methods with quite different computational efficiency and convergence properties.

Moh'd et al., (2014) presents a Hybrid Broyden-Fletcher-Goldfab-Shanno (HBFGS) method which used the search direction of the conjugate gradient methods with quasi-Newton update where their numerical result provides strong evidence that the proposed HBFGS method is more efficient than the ordinary BFGS method. Furthermore, Mustapha et al. (2014) followed the approach in Mohammad, (2014) to present a Hybrid BFGS CG method for solving unconstrained optimization problems; the method has been presented based on combining search directions between conjugate gradient method and quasi-Newton method where the methods have shown some significant improvement for solving large-scale problems with less number of iterations and CPU time respectively. However, Hamed et al. (2019) equally modified the work in Mohammad et al (2014) and Mustapha et al. (2014) and presented a new algorithm for convex Nonlinear Unconstrained optimization problems by proposing the search direction as defined in Mohammad et al (2014) and Mustapha et al. (2014). The new search direction is defined as:

$$d_{k+1} = -\lambda_k g_{k+1} + \rho_k H_{k+1} g_{k+1} \tag{4}$$

Where H_{k+1} is the approximation matrix of BFGS updated matrix and λ_k is a positive constant. And the parameters λ_k and ρ_k are respectively defined as:

$$\lambda_k = \frac{(1+t)s_k^T g_{k+1}}{y_k^T g_{k+1}}, \tag{5}$$

and

$$\rho_k = \frac{\lambda_k y_k^T g_{k+1} - t s_k^T g_{k+1}}{s_k^T g_{k+1}}. \tag{6}$$

Where $t > 0$ and the conjugate gradient (CG) parameter is obtained as:

$$\beta_k = \frac{\psi_k \rho_k y_k^T g_{k+1} + s_k^T y_k g_{k+1} s_k^T - \lambda_k s_k^T y_k g_{k+1}^T s_k}{||s_k||^2 s_k^T y_k} \tag{7}$$

Therefore, it is very important to state that, solving BFGS-CG methods is severally used in unconstrained optimization problems, they are particularly efficient due to their rapid Convergence properties, simple to implement and low storage requirement Waziri, (2015) and Hamed, (2019). However, they are very scanty in solving system of nonlinear equations, this is what motivated us to write this paper. Furthermore, (1) can come from an unconstrained optimization problem, a saddle point and equality constrained problem Li, (1999). Let f be a norm function defined by;

$$f(x) = \frac{1}{2}||F(x)||^2. \tag{8}$$

Then the nonlinear equation in problem (1) is equivalent to the following global optimization problem Waziri, (2015) and Sun, (2006).

$$min f(x), \quad x \in \mathbb{R}^n. \tag{9}$$

We organize the paper as follows. In the next section, we present the details of our proposed method, convergence result is presented in section 3. Some numerical results are reported in section 4. Finally, conclusions are made in section 5.

MATERIALS AND METHODS

Derivation of the method

In this section, we present new hybrid conjugate gradient (CG) update parameter β_k, via two other parameters λ_k and ρ_k. This is made possible by combining the search direction proposed by Hamed et al. (2019), given by:

$$d_{k+1} = -\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}), \tag{10}$$

together with the classical Newton direction given by:

$$d_{k+1} = -J_{k+1}^{-1} F(x_{k+1}) + \lambda_k d_k. \tag{11}$$

Where J_{k+1}^{-1} is the inverse Jacobian matrix. Multiplying equation (10) and (11) by y_k^T we have;

$$y_k^T d_{k+1} = -\lambda_k y_k^T F(x_{k+1}) + \rho_k y_k^T J_{k+1}^{-1} F(x_{k+1}), \tag{12}$$

and

$$y_k^T d_{k+1} = -y_k^T J_{k+1}^{-1} F(x_{k+1}) + \lambda_k y_k^T d_k. \tag{13}$$

By conjugacy condition;

$$y_k^T d_{k+1} = 0\,. \tag{14}$$

And also from weak secant condition, i.e

$$y_k^T J_{k+1}^{-1} = s_k^T. \tag{15}$$

We assume J_{k+1}^{-1} is symmetric. By applying (14) and (15) in (12) we obtain:

$$\lambda_k = \frac{s_k^T F(x_{k+1})}{y_k^T d_k}. \tag{16}$$

Also, substituting equations (14) and (15) in (12) we have;

$$\rho_k = \frac{\lambda_k y_k^T F(x_{k+1})}{s_k^T F(x_{k+1})} \tag{17}$$

Recall that the classical CG direction is defined to obtain an updated version of the conjugate gradient method associated with new parameter β_k, we compare the standard CG direction;

$$d_{k+1} = -F(x_{k+1}) + \beta_k s_k, \tag{18}$$

with

$$d_{k+1} = -\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}). \tag{19}$$

Therefore, form (18) and (19) we have;

$$-\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}) = -F(x_{k+1}) + \beta_k s_k. \tag{20}$$

Multiplying (20) by y_k^T we have;

$$-\lambda_k y_k^T F(x_{k+1}) + \rho_k y_k^T J_{k+1}^{-1} F(x_{k+1}) = -y_k^T F(x_{k+1}) + \beta_k y_k^T s_k \tag{21}$$

After some algebraic simplifications, we obtain our proposed parameter β_k as;

$$\beta_k = \frac{\rho_k s_k^T F(x_{k+1}) + y_k^T F(x_{k+1}) - \lambda_k y_k^T F(x_{k+1})}{y_k^T s_k}. \tag{22}$$

(22) further simplifies to:

$$\beta_k = \frac{\rho_k s_k^T F(x_{k+1}) + (1-\lambda_k) y_k^T F(x_{k+1})}{y_k^T s_k}. \tag{23}$$

Finally, substituting (16) and (17) in (23), our CG parameter β_k becomes:

$$\beta_k = \frac{y_k^T F(x_{k+1})}{y_k^T s_k}, \tag{24}$$

and our search direction is given by:

$$d_{k+1} = -F(x_{k+1}) + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right) s_k \quad (25)$$

Remark 1

For our search direction to satisfy the sufficient decent condition;

$$F_k^T d_k \le -c||F_k||^2, \quad c>0 \quad (26)$$

we re-define our search direction as follows;

$$d_{k+1} = -\theta F(x_{k+1}) + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right) s_k, \quad (27)$$

where θ is a parameter to be determined in such a way the search direction satisfies the decent condition in (26). Multiplying equation (27) by F_{k+1}^T gives:

$$F_{k+1}^T d_{k+1} = -\theta||F_{k+1}||^2 + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right) F_{k+1}^T s_k$$
$$\le -\theta||F_{k+1}||^2 + |\frac{y_k^T F(x_{k+1})}{y_k^T s_k}|\,||F_{k+1}^T s_k| \quad (28)$$
$$\le -\theta||F_{k+1}||^2 + \frac{||y_k||\,||F_{k+1}||^2}{y_k^T s_k}||s_k||$$

The second inequality follows from Cauchy-Schwartz inequality. But $m||s_k|| \le ||y_k|| \le M||s_k||$ (see Lemma 3.2 of [9]), i.e $y_k^T s_k \ge m\big||s_k|\big| = \frac{1}{y_k^T s_k} \le \frac{1}{m||s_k||^2}$ $for\ M > m > 0.$

Inequality (28) implies that:

$$F_{k+1}^T d_{k+1} \le -\theta||F_{k+1}||^2 + \frac{M||s_k||^2}{m||s_k||^2}||F_{k+1}||^2$$
$$\le -\theta||F_{k+1}||^2 + \frac{M}{m}||F_{k+1}||^2 \quad (29)$$
$$\le -\left(\theta - \frac{M}{m}\right)||F_{k+1}||^2.$$

For the search direction to satisfy (26), we need;

$$\theta \ge c + \frac{M}{m}, \quad (30)$$

where c is a positive constant. Without loss of generality, we select:

$$\theta = c + \frac{M}{m}. \quad (31)$$

Hence, the inequality in (29) becomes;

$$F_{k+1}^T d_{k+1} \le -\left(c + \frac{M}{m} - \frac{M}{m}\right)||F_{k+1}||^2 = -c||F_{k+1}||^2. \quad (32)$$

Which clearly shows that, our conjugate gradient search direction satisfies the sufficient decent condition in (26).

Furthermore, to compute the step-length α_k, we apply the derivative-free line search procedure proposed by Li and Fukushima in Li, (1999).

Let $\omega_1 > 0,\ \omega_2 > 0\ and\ r \in (0,1)$ be constants and let $\{\eta_k\}$ be a given positive sequence such that:

$$\sum_{k=0}^{\infty} \eta_k < \eta < \infty \quad (33)$$

$$f(x_k + \alpha_k d_k) - f(x_k) \le -\omega_1\big||\alpha_k F(x_k)|\big|^2 - \omega_2\big||\alpha_k d_k|\big|^2 + \eta_k f(x_k), \quad (34)$$

where $\alpha_k = r^{i_k}$ and i_k is the smallest non-negative integer i such that (34) holds with α_k replaced by r^{i_k}.

We can describe the algorithm of our method as follows:

Algorithm 1: (Spectral Hybrid Conjugate Gradient Algorithm)

tep 1: Given $x_0 \in \mathbb{R}^n,\ \varepsilon > 0, d_0 = -F(x_0),\ set\ \ k = 0.$
Step 2: Compute $F(x_k)$.
Step 3: If $||\,F(x_k)|| \le \varepsilon,\ then\ stop,\ else,\ go\ to\ step\ 4.$
Step 4: Compute the step-length α_k using (34).
Step 5: Set $x_{k+1} = x_k + \alpha_k d_k$.
Step 6: Compute $F(x_{k+1})$.
Step 7: Compute $y_k = F(x_{k+1}) - F(x_k)$.
Step 8: Compute $\lambda_k\ and\ \rho_k$ from (16) and (17).
Step 9: Compute β_k using (23).
Step 10: Update d_{k+1} using (27).
Step 11: Set $k = k + 1$ and go back to step 2.

Convergence Analysis

This section is devoted to a study of the global convergence of our method (SHCGA). To begin with, let us define the level set:

$$\Omega = \{x : \big||F(x)|\big| \le \big||F(x_0)|\big|\} \quad (35)$$

The following basic assumptions are required in order to analyze the convergence of our algorithm 1.

Assumptions:

There exists $x^* \in \mathbb{R}^n$ such that $F(\,x^*) = 0$.

F is continuously differentiable in a neighborhood of x^*.

The level set Ω as defined by (35) is bounded.

CG direction is a good approximation to Newton direction, i.e

$$\big||F'(x_{k+1})d_{k+1} - (d_{k+1} - \beta_k s_k)|\big| \le \varepsilon\big||F(x_{k+1})|\big|, \quad (36)$$

where $\varepsilon\ \epsilon (0,1)$ is a small quantity [12], [22] and [24].

The Jacobian of F is bounded and positive definite on N. i.e there exists positive constants $M > m > 0$ such that:

$$\big||F'(x)|\big| \le M \qquad \forall x \epsilon N,\ and \quad (37)$$
$$m\big||d^2|\big| \le d^T F'(x) d \qquad \forall x\ x\epsilon N, \quad d\epsilon\mathbb{R}^n. \quad (38)$$

Lemma 1 Suppose assumption 1 holds. Let $\{x_k\}$ be generated by the SHCGA algorithm, then

$$\lim_{k\to\infty} ||\alpha_k d_k\,|| = \lim_{k\to\infty}\big||s_k|\big| = 0, \quad (39)$$

and

$$\lim_{k\to\infty} ||\alpha_k F(x_k)\,|| = 0 \quad (40)$$

Proof: From the line search in equation (34) and for all $k > 0,$ we obtain:

$$\omega_2\big||\alpha_k d_k|\big|^2 \le \omega_1\big||\alpha_k F_k|\big|^2 + \omega_2\big||\alpha_k d_k|\big|^2$$
$$\le ||F_k||^2 - ||F_{k+1}||^2 + \eta_k||F_k||^2. \quad (41)$$

And by summing up the above k inequality, we obtain:

$$\omega_2 \sum_{i=0}^{k}\big||\alpha_k d_k|\big|^2 \le \sum_{i=0}^{k}\left(\big||F(x_i)|\big|^2 - \big||F(x_{i+1})|\big|^2\right) + \sum_{i=0}^{k}\eta_i||F(x_i)^2$$
$$= ||F(x_0)||^2 - ||F(x_{k+1})||^2 + \sum_{i=0}^{k}\eta_k||F(x_i)||^2$$
$$\le ||F(x_0)||^2 + ||F(x_0)||^2\sum_{i=0}^{k}\eta_i$$
$$\le ||F(x_0)||^2 + ||F(x_0)||^2\sum_{i=0}^{\infty}\eta_i$$
$$\le M^2 + M^2\sum_{i=0}^{\infty}\eta_i \quad (42)$$

Therefore, from the level set and the fact that $\{\eta_k\}$ satisfies (33), then the series $\sum_{i=0}^{k}||\alpha_k d_k||^2$ is convergent, which implies that (39) holds. using the same argument as above, with $\omega_1\left||\alpha_k F_k|\right|^2$ on the left hand side, we obtain (40).

Lemma 2 Suppose assumption 1 holds. Let the sequence $\{x_k\}$ be generated by the SHCGA algorithm with update parameter β_k, then there exists a constant $m_2 > 0$ such that for k>0,

$$\left|\left|d_k^{SHCGA}\right|\right| \leq m_2 \quad (43)$$

Proof: From (27) we have

$$\left||d_{k+1}|\right| = |\left|-\theta F(x_{k+1}) + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right)s_k\right||. \quad (44)$$

Applying triangular inequality we've;

$$\left||d_{k+1}|\right| \leq |\theta|\left||F(x_{k+1})|\right| + \left|\left|\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right|\right| ||s_k|| \quad (45)$$

$$\leq |\theta|\left||F(x_{k+1})|\right| + \frac{|y_k^T F(x_{k+1})|||s_k||}{y_k^T s_k} \quad (46)$$

$$\leq |\theta|\left||F(x_{k+1})|\right| + \frac{||y_k|||F(x_{k+1})|||s_k||}{y_k^T s_k} \quad (47)$$

$$\leq |\theta|||F(x_{k+1}) + \frac{||y_k|||F(x_{k+1})|||s_k||}{m||s_k||^2} \quad (48)$$

$$\leq |\theta|\left||F(x_{k+1})|\right| + \frac{M||s_k||^2||F(x_{k+1})||}{m||s_k||^2} \quad (49)$$

Inequality (46) follows from Cauchy-Schwartz inequality. From the level set and (31) we have;

$$\left||d_{k+1}|\right| \leq \left(c + \frac{M}{m}\right)\left||F(x_0)|\right| + \frac{M||F(x_0)||}{m} \quad (50)$$

$$\leq \left||F(x_0)|\right| + \frac{M||F(x_0)||}{m} \quad (51)$$

$$\leq \left(c + \frac{2M}{m}\right)\left||F(x_0)|\right| = m_2. \quad (52)$$

Therefore, (52) shows that (43) holds.

We are now going to establish the global convergence of our method, in order to show that under some suitable conditions, there exists an accumulation point of sequence x_k which is a solution of problem (1).

Theorem: Suppose assumption 1 holds and that the sequence $\{x_k\}$ is generated by the SHCGA algorithm. Also, assume that for all k>0,

$$\alpha_k \geq c\frac{|F(x_k)^T d_k|}{||d_k||^2}, \quad (53)$$

where c is some positive constant. Then $\{x_k\}$ converges globally to a solution of problem (1); i.e.,

$$\lim_{k\to\infty}\left||F(x_k)|\right| = 0. \quad (54)$$

Proof: By the boundedness of d_k, we have;

$$\lim_{k\to\infty}\alpha_k||d_k||^2 = 0. \quad (55)$$

From (53) and (55) we have

$$\lim_{k\to\infty}|F(x_k)^T d_k| = 0. \quad (56)$$

From our proposed direction, we have;

$$d_{k+1} = -F(x_{k+1}) + \beta_k s_k \quad (57)$$

Therefore, by multiplying (57) by $F(x_{k+1})^T$, we obtain:

$$F(x_{k+1})^T d_{k+1} = -F(x_{k+1})^T F(x_{k+1}) + \beta_k F(x_{k+1})^T s_k. \quad (58)$$

$$||F(x_{k+1})||^2 = -F(x_{k+1})^T d_{k+1} + \beta_k F(x_{k+1})^T s_k. \quad (59)$$

$$||F(x_{k+1})||^2 \leq |-F(x_{k+1})^T d_{k+1}| + |\beta_k F(x_{k+1})^T s_k|. \quad (60)$$

$$||F(x_{k+1})||^2 \leq |-F(x_{k+1})^T d_{k+1}| + |\beta_k||| F(x_{k+1})^T|||\left|s_k|\right|. \quad (61)$$

But (53) implies that;

$$\alpha_k||d_k||^2 \geq c|F(x_k)^T d_k|, \quad (62)$$

Since $||d_k||$ is bounded and $\lim_{k\to\infty}|F(x_k)^T d_k| = 0$, we have $\lim_{k\to\infty}\alpha_k||d_k||^2 = 0$. Thus,

$$0 \leq c|F(x_k)^T d_k| \leq \alpha_k\left||d_k|\right|^2 \to 0. \quad (63)$$

Then we have;

$$||F(x_k)||^2 \leq |-F(x_k)^T d_k| + |\beta_k|\left||F(x_k)^T|\right|\left||s_k|\right| \to 0. \quad (64)$$

Therefore,

$$\lim_{k\to\infty}\alpha_k||d_k||^2 = 0. \quad (65)$$

The proof is completed.

RESULTS AND DISCUSSION

Numerical Results

In this section, the performance of our method for solving systems of nonlinear equations compared with NHCG method for symmetric nonlinear equations [1] and NHCGP [18] is reported.

SHCGA stands for our method and both cases, we set the following:

$\omega_1 = \omega_2 = 10^{-4},\ r = 0.2.$

The codes were written in MATLAB 8.9.0 (R2014a) and run on a personal computer 2.00GHz CPU processor and 3GB RAM memory. We stopped the iterations if the total number of iterations exceeds 1000 or $||F(x_k)|| \leq 10^{-4}$. We tested the three methods using twenty two (22) test problems with different initial starting points (x_0), and dimensions (n-values). We present here some of the benchmark test problems with dimensions 1,000, 10,0000, 20,000, 50,000 and 100,000 respectively used to test our proposed method in this research (i.e SHCGA).

Table 1: The Summary of Numerical Results

	ALGORITHMS			
	SHCGA	**NHCG**	**NHCGP**	**Undecided**
Total number of problems	120	120	120	
Total number of problems	110	110	110	
Problems solved with less number of iterations	95	10	12	13
Percentage	77.27%	4.54%	6.36%	11.83%
Problems solved with less CPU time	90	10	8	12
Percentage	72.72%	9.09%	7.27%	10.92%

To illustrate the performance of the three methods, a summary of the results is presented in table1. The summarized data shows the number of problems for which method is a winner, in terms of number of iterations and CPU time respectively. The corresponding percentages of number of problems solved by each method are also reported. The summary reported in table 1 indicates that the SHCGA scheme is a winner with respect to number of iterations and CPU time. The table shows that, the SHCGA method solves 77.27% (95 out of 120) of the problems with less number of iterations, compared to the NHCG method, which solves 4.54% (10 out of 120)
and NHCGP method which solves 6.36% (12 out of 120). The summarized result also shows that both methods solve 13 problems with the same number of iterations, which translates to 11.83% and is reported as undecided. Also, the summary indicates that the SHCGA scheme outperforms
the NHCG and NHCGP methods as it solves 72.72% (90 out of 120) of the problems with less CPU time compared to 9.09% (10 out of 120) solved by the NHCG method and 7.27% (8 out of 120) by the NHCGP method. Therefore, it is evident from figures 1 and 2 and the summarized result in table 1 that, our method is more effective than the NHCG and NHCGP methods, and therefore, more suitable for solving large-scale system of nonlinear equations.

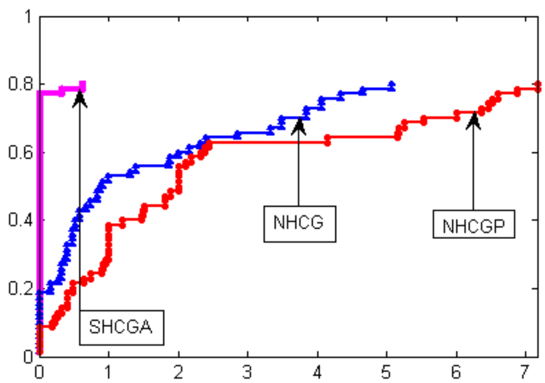

Figure 1: Performance profile of SHCGA, NHCG and NHCGP Algorithms with respect to the number of iterations for the problems

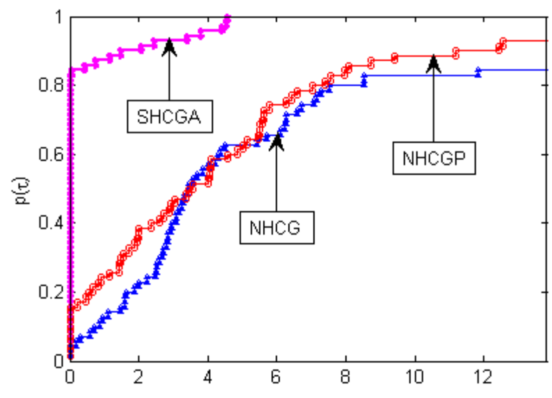

Figure 2: Performance profile of SHCGA, NHCG and NHCGP Algorithms with respect to the CPU time (in seconds) for the problems

Figures (1) and (2) show the performance of our method based on the number of iterations and CPU time respectively, which were evaluated using the profiles of Dolan, (2001). For each method, we plot the fraction $p(\tau)$ of the problems for which the method is within a factor τ of the best time. The top curve is the method that solved the most problems in a time that was within a factor τ of the best time. The summary of the numerical results of the three (3) methods are reported in Table 1. The summary of numerical results indicates that the proposed method, i.e SHCGA has minimum number of iterations and CPU time, compared to NHCG and NHCGP respectively. Except for problems 1 and 11 where the number of iterations in SHGCA of large dimension is more than that of NHCG and NHCGP. We can easily see that our claim is fully justified from the table, that is, less CPU time and number of iterations for each test problem with the exception of problems 1 and 11. Furthermore, on the average, our $||F(x_k)||$ is too small which signifies that the solution obtained is the true approximation of the exact solution compared to NHCG and NHCGP schemes.

CONCLUSION

In this paper, we presented a new spectral hybrid conjugate gradient algorithm (SHCGA), for solving large-scale system of nonlinear equations and compared its performance with that of (NHCG and NHCGP) methods for symmetric nonlinear equations by performing some numerical experiments. We however proved the global convergence of our proposed method, using a derivative-free line search proposed by Li and Fukushima, and the numerical results show that our method is very effective. This research can be extended to large-scale nonlinear monotone system of equations with applications to signal and image recovery problems.

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