



Mathematical Analysis of Gravitational Motion and Orbital Stability in a Two-Body System

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ABSTRACT

The Newtonian two-body problem provides an analytical basis for relating gravitational attraction to orbital geometry and stability. This paper presents a unified treatment of relative motion under an inverse-square central force, using the centre of mass transformation and a consistent set of orbital parameters. The Lagrangian formulation separates translational and relative motion, while conservation of angular momentum and mechanical energy reduces the dynamics to an effective radial potential. Circular equilibria are examined through the first and second derivatives of this potential, establishing the condition for local radial stability at fixed angular momentum. Transformation to the reciprocal radial coordinate produces Binet's equation, whose Newtonian solution yields conic trajectories classified by eccentricity and specific orbital energy. Conservation of areal velocity also leads to the orbital period relation expressed by Kepler's third law. The planetary quantities reproduced in the accompanying table illustrate this established proportionality and are not independent observations. The analysis shows how the main results of classical orbital mechanics follow from a common mathematical framework, while distinguishing the local stability of the idealized two-body orbit from long-term behaviour under additional gravitational and nonconservative perturbations.

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INTRODUCTION

The study of planetary and satellite motion depends on understanding the relationship between orbital patterns and the gravitational forces responsible for them. Kepler's description of planetary trajectories established the geometric principles governing orbital motion (Agol & Fabrycky, 2017). Newtonian gravitation subsequently provided a physical explanation of these principles through the inverse-square law of gravitational attraction (Dombouya, 2026). Although gravitational interactions among several celestial bodies often require complex mathematical approaches, the motion of two isolated bodies can be described analytically (Jesus, 2026). Consequently, the two-body problem remains

fundamental to orbital prediction, spacecraft trajectory analysis and celestial mechanics (Musielak & Quarles, 2014). It also provides a reference model for examining orbital deviations arising from nonspherical gravitational fields (Gurfil & Efroimsky, 2021; Saillenfest, 2017).

The mathematical treatment of the two-body problem involves separating the motion of the centre of mass from the relative motion produced by gravitational interaction (Gómez et al., 2001; Qiao et al., 2025). Within the Lagrangian formulation, this separation is achieved through generalized coordinates and the identification of conserved quantities associated with cyclic coordinates (Bini & Damour, 2012). Conservation of angular momentum and mechanical energy allows the governing

equations to be expressed in a reduced form, simplifying the analysis of orbital behaviour. The effective potential combines the angular momentum contribution with gravitational potential energy and provides a means of examining radial motion and equilibrium conditions (Britt, 2021; Schumayer & Hutchinson, 2019). Analytical treatments of orbital mechanics have demonstrated the relationship between these conservation principles, orbital stability and Keplerian trajectories (Krishnaswami & Senapati, 2019). Similar formulations establish the connection between central gravitational forces and the geometric properties of orbital paths (Samboy & Gallant, 2024).

For motion governed by a central force, conservation of angular momentum restricts the trajectory to a fixed plane and ensures that the radius vector sweeps equal areas during equal time intervals (Middleton, 2015; Samboy & Gallant, 2024). The effective potential formulation also provides a criterion for determining the stability of circular orbits. A circular equilibrium is locally stable against sufficiently small radial disturbances at fixed angular momentum when the effective potential has a strict local minimum. However, this condition does not establish stability under arbitrary external disturbances or changing orbital parameters (Adams & Bloch, 2014; Morey, 2023). Realistic gravitational systems may involve additional forces and interactions that cannot be represented by an isolated central-force model (Celletti et al., 2023; Galeş et al., 2025). The classical two-body solution is therefore commonly used as a reference for understanding more complex orbital behaviour rather than as a complete description of all celestial motion (Marsden & Ross, 2005). Systems involving multiple gravitationally interacting bodies may exhibit additional dynamical structures and stability characteristics (Nazari et al., 2023).

The continued relevance of the two-body approximation lies in its ability to establish a mathematically consistent foundation for investigating orbital dynamics (Brzeski & Perlikowski, 2018). The centre of mass transformation and reduced-mass formulation simplify the equations governing relative motion while preserving the essential gravitational interaction (Conway & Will, 2025). Nevertheless, the distinction between local orbital stability and the long-term evolution of gravitational systems becomes particularly important when resonances and secular perturbations are considered (Lei et al., 2018). Restricted three-body models demonstrate dynamical behaviour that cannot be described entirely through conventional Keplerian trajectories (Lei, 2019). In practical orbital analysis, perturbation methods and numerical propagation techniques are therefore employed when the assumptions of the ideal two-body model are insufficient (Tos & Topputo, 2017).

An alternative analytical description of gravitational motion is obtained through Binet's equation, which

expresses the orbital radius as a function of the polar angle. For Newtonian gravitational attraction, the resulting solution describes a conic trajectory whose geometry depends on orbital eccentricity and mechanical energy. Circular and elliptical trajectories represent bound motion, while parabolic and hyperbolic trajectories correspond to limiting and unbound cases, respectively. The same analytical framework connects conservation of areal velocity with the relationship between orbital period and semi-major axis expressed by Kepler's third law (Ramond & Perez, 2020). In practical orbit determination, deviations from ideal Keplerian elements may arise from noncentral accelerations and observational constraints (Baù et al., 2015).

The assumptions of the two-body model also exclude the rotational dynamics of extended celestial bodies and spacecraft. Such systems require additional equations to describe attitude motion and rotational stability (Nazari et al., 2023). More complex mission designs may involve periodic trajectories and invariant manifolds that cannot be represented by a single Keplerian orbit (Bucci et al., 2018). Similarly, investigations of central configurations address gravitational equilibria and geometric arrangements beyond the isolated two-body system (Zhang et al., 2015). These developments illustrate the broader applicability of celestial mechanics while defining the conditions under which the classical orbital approximation remains appropriate.

The influence of extended-body dynamics provides a further distinction between ideal gravitational motion and more realistic celestial systems. Models involving celestial-scale dumbbells account for the coupling between translational and rotational motion (Bini et al., 2020). Their orbital behaviour may include periodic solutions and dynamical interactions absent from the point-mass approximation (Scheeres, 2004). Investigations of tidal interactions have also demonstrated that gravitational effects can influence the coupled dynamics of extended bodies (Crespo et al., 2024). Tidal forces arise from spatial variations in gravitational attraction and may produce responses that cannot be described by a simple central-force potential (Lee, 2008). More detailed treatments of tidal phenomena consequently require additional assumptions concerning gravitational interactions and physical properties (Correia et al., 2014; Lu et al., 2023). Viscoelastic tidal models further account for deformation and energy dissipation, extending classical orbital descriptions to systems with more complex physical behaviour (Remus et al., 2012).

The reviewed literature establishes the analytical completeness of the ideal Newtonian two-body problem while identifying the limitations associated with more complex gravitational systems. The classical formulation provides consistent relationships among reduced mass, conservation laws, effective potential, orbital stability,

conic trajectories and orbital periods. In contrast, perturbations, additional degrees of freedom and dissipative interactions require mathematical treatments beyond the isolated two-body approximation. Against this background, the present study examines the Newtonian two-body problem through a consistent analytical formulation, with particular attention to circular-orbit stability, orbital classification and Kepler's laws. The analysis develops these established relationships through explicit mathematical derivations while recognizing the

limitations of the model in describing more complicated orbital systems.

REVIEW METHODOLOGY

The notation used throughout the derivation is summarized in Tables 1 and 2. Scalar symbols are used wherever possible. The standard gravitational parameter is denoted by μ , while m_r denotes the reduced mass; these quantities have different physical meanings and are not interchangeable.

Table 1: Physical Quantities and Proposed Mathematical Notation

Physical quantity	Notation
Universal gravitational constant	G
Mass of the first body	M_1
Mass of the second body	M_2
Total mass of the system	M_t
Reduced mass	m_r
Standard gravitational parameter	μ
Gravitational interaction constant	K
Gravitational force	F
Radial gravitational force	F_r
Relative radial distance	r
Centre of mass coordinate	R_c
Orbital polar angle	θ
Angular velocity	$\dot{\theta}$
Radial velocity	$\dot{r}$
Radial acceleration	$\ddot{r}$
Lagrangian function	$\mathcal{L}$
Angular momentum magnitude	L
Specific angular momentum	h
Gravitational potential energy	$V(r)$
Effective potential energy	$V_e(r)$
Total mechanical energy	E
Specific orbital energy	ε
Circular orbit radius	r_c
Reciprocal radial coordinate	u
Orbital eccentricity	e
Semi-major axis	a
Semi-minor axis	b
Semi-latus rectum	p
Periapsis distance	r_p
Apoapsis distance	r_a
Orbital period	T
Mean orbital motion	n
Swept orbital area	$\mathcal{A}$

Table 2: Definitions of Principal Mathematical Parameters

Notation	Mathematical definition
M_t	$M_1 + M_2$
m_r	$M_1 M_2 / (M_1 + M_2)$
μ	$G(M_1 + M_2)$
K	$GM_1 M_2$
L	$m_r r^2 \dot{\theta}$
h	$L / m_r = r^2 \dot{\theta}$

$V(r)$	$-K/r$
$V_e(r)$	$L^2/(2m_r r^2) - K/r$
ε	E/m_r
r_c	h^2/μ
u	$1/r$
p	h^2/μ
b	$a\sqrt{1 - e^2}$
r_p	$a(1 - e)$
r_a	$a(1 + e)$
T	$2\pi\sqrt{a^3/\mu}$
n	$\sqrt{\mu/a^3}$

The expressions for b , r_p , r_a , T , and n apply to bound elliptical Keplerian orbits. The circular radius expression applies to Newtonian circular motion.

Mathematical Model and Central Gravitational Force

Consider two bodies with masses M_1 and M_2 and relative separation r . Under Newtonian gravitation, the magnitude of their mutual attraction varies inversely with the square of the separation. Define $K = GM_1M_2$ and $\mu = G(M_1 + M_2)$.

With the radial direction taken as positive outwards, the gravitational force is negative because it points towards the attracting body.

$$F = \frac{GM_1M_2}{r^2} = \frac{K}{r^2} \quad (1)$$

$$M_t = M_1 + M_2 \quad (2)$$

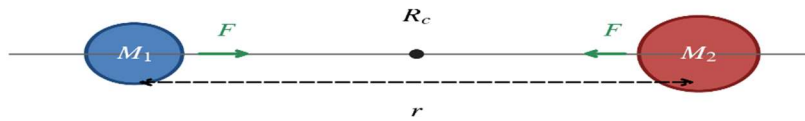


Figure 1: Schematic representation of the gravitational interaction between two bodies with masses

M_1 and M_2 , separated by distance r , showing the gravitational forces and centre of mass coordinate R_c . Figure 1 illustrates the gravitational interaction between two bodies of masses M_1 and M_2 , separated by a radial distance r . Their mutual gravitational forces act along the line connecting the two bodies and are directed towards one another, in accordance with Newton's law of universal gravitation. The force magnitude is $F = GM_1M_2/r^2$, while R_c identifies the centre of mass of the system. The schematic defines the physical configuration used in the relative equations of motion and the Lagrangian formulation.

Centre of Mass and Lagrangian Formulation

Let the individual position coordinates along the line of relative separation be x_1 and x_2 , and define the relative coordinate $r = x_2 - x_1$. The centre of mass coordinate is the mass-weighted average of the two positions. The total mass and reduced mass are defined as follows.

$$M_t = M_1 + M_2 \quad (3)$$

$$m_r = \frac{M_1M_2}{M_t} \quad (4)$$

$$R_c = \frac{M_1x_1 + M_2x_2}{M_t} \quad (5)$$

Solving the centre of mass and relative-coordinate definitions gives the positions relative to the centre of

mass. These relations separate translational motion of the system from internal relative motion.

$$x_1 = R_c - \frac{M_2}{M_t}r \quad (6)$$

$$x_2 = R_c + \frac{M_1}{M_t}r \quad (7)$$

Substitution of these expressions into the total kinetic energy cancels the cross terms. The kinetic energy separates into centre of mass motion and relative motion, where the coefficient of the relative kinetic energy is the reduced mass.

$$\mathcal{T} = \frac{1}{2}M_t\dot{R}_c^2 + \frac{1}{2}m_r\dot{r}^2 \quad (8)$$

For planar relative motion, the squared relative speed is $\dot{r}^2 + r^2\dot{\theta}^2$. The gravitational potential energy is $V(r) = -K/r$. Therefore, the Lagrangian for the relative coordinate is the kinetic energy minus the potential energy.

$$\mathcal{L} = \frac{1}{2}m_r(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{K}{r} \quad (9)$$

Applying the Euler-Lagrange equation to r gives the radial equation of motion. Differentiation with respect to $\dot{r}$ gives $m_r\dot{r}$, while differentiation of the Lagrangian with respect to r gives $m_r r\dot{\theta}^2 - K/r^2$.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \quad (10)$$

$$m_r\ddot{r} - m_r r\dot{\theta}^2 + \frac{K}{r^2} = 0 \quad (11)$$

The angle θ is a cyclic coordinate because the Lagrangian does not depend explicitly on θ . Its conjugate momentum is consequently constant.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \quad (12)$$

$$\frac{d}{dt} (m_r r^2 \dot{\theta}) = 0 \quad (13)$$

Angular momentum and mechanical energy

The conserved angular momentum magnitude and the specific angular momentum are defined by the constant conjugate momentum and its value per unit reduced mass.

$$L = m_r r^2 \dot{\theta} = \text{constant} \quad (14)$$

$$h = \frac{L}{m_r} = r^2 \dot{\theta} \quad (15)$$

The area swept out in a short time interval is approximately one half of the product of r and the transverse displacement $r d\theta$. Dividing by the time interval and taking the limit yields the areal velocity.

$$\frac{d\mathcal{A}}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{h}{2} \quad (16)$$

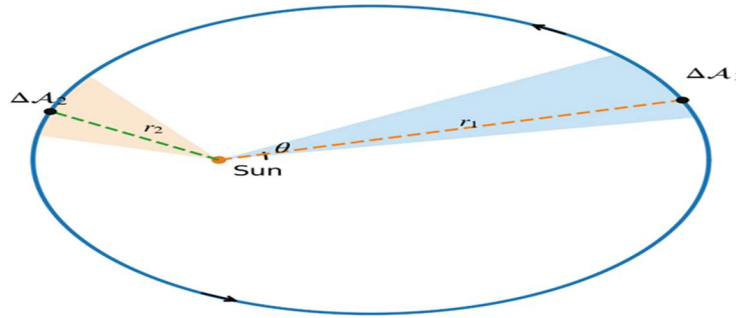


Figure 2: Elliptical orbital motion illustrating Kepler's second law and the relationship between radial displacement, angular position, and swept orbital area

Figure 2 illustrates elliptical motion about a central mass situated at a focus of the orbit. The shaded sectors $\Delta\mathcal{A}_1$ and $\Delta\mathcal{A}_2$ represent areas swept by the radius vector over two orbital intervals. Conservation of specific angular momentum, $h = r^2 \dot{\theta}$, gives the constant areal velocity $d\mathcal{A}/dt = h/2$. Consequently, equal areas are swept in equal times, although orbital speed varies with distance: motion is faster near periapsis and slower near apoapsis. The shaded regions are schematic, not a numerical demonstration of equal areas.

The total mechanical energy is conserved because the Newtonian gravitational force is conservative. In polar coordinates, the relative kinetic energy contains radial and angular components. Replacing $\dot{\theta}$ with $L/(m_r r^2)$ yields a one-dimensional radial energy equation.

$$E = \frac{1}{2} m_r \dot{r}^2 + \frac{1}{2} m_r r^2 \dot{\theta}^2 - \frac{K}{r} \quad (17)$$

$$E = \frac{1}{2} m_r \dot{r}^2 + \frac{L^2}{2m_r r^2} - \frac{K}{r} \quad (18)$$

$$V_e(r) = \frac{L^2}{2m_r r^2} - \frac{K}{r} \quad (19)$$

$$E = \frac{1}{2} m_r \dot{r}^2 + V_e(r) \quad (20)$$

Stability of circular orbits

A circular orbit has constant radius, so $\dot{r} = 0$ and the radial acceleration vanishes. Its radius is determined by a stationary point of the effective potential. A local minimum indicates stability against sufficiently small radial perturbations when angular momentum is held fixed.

$$\frac{dV_e}{dr} = -\frac{L^2}{m_r r^3} + \frac{K}{r^2} \quad (21)$$

$$\frac{dV_e}{dr} = 0 \Rightarrow r_c = \frac{L^2}{m_r K} \quad (22)$$

Using $h = L/m_r$ and $K = m_r \mu$, the circular radius can also be expressed in terms of specific angular momentum and the standard gravitational parameter.

$$r_c = \frac{h^2}{\mu} \quad (23)$$

$$\frac{d^2 V_e}{dr^2} = \frac{3L^2}{m_r r^4} - \frac{2K}{r^3} \quad (24)$$

At the circular radius, $L^2 = m_r K r_c$. Substitution into the second derivative gives a positive curvature for $K > 0$, confirming a local minimum for the Newtonian attractive potential.

$$\left. \frac{d^2 V_e}{dr^2} \right|_{r=r_c} = \frac{K}{r_c^3} > 0 \quad (25)$$

Derivation of Binet's equation

To express the trajectory as a function of angle rather than time, introduce the reciprocal radial coordinate $u = 1/r$. Conservation of angular momentum gives $\dot{\theta} = h/r^2 = hu^2$. Differentiating $r = 1/u$ with respect to time then relates the radial velocity to the angular derivative of u .

$$u = \frac{1}{r}, \quad \dot{\theta} = hu^2 \quad (26)$$

$$\dot{r} = \frac{dr}{d\theta} \dot{\theta} = -h \frac{du}{d\theta} \quad (27)$$

Differentiating once more with respect to time, while using $\dot{\theta} = hu^2$, produces the radial acceleration in terms of u and its second derivative with respect to θ .

$$\ddot{r} = -h^2 u^2 \frac{d^2 u}{d\theta^2} \quad (28)$$

The radial equation is $m_r(\ddot{r} - r\dot{\theta}^2) = F_r$. Substituting $r = 1/u$, $\dot{\theta} = hu^2$, and the expression for $\ddot{r}$ gives Binet's equation.

$$\frac{d^2u}{d\theta^2} + u = -\frac{F_r(1/u)}{m_r h^2 u^2} \quad (29)$$

For Newtonian gravitation, $F_r = -K/r^2 = -Ku^2$ and $K = m_r\mu$. The force term therefore reduces to a constant.

$$\frac{d^2u}{d\theta^2} + u = \frac{K}{m_r h^2} = \frac{\mu}{h^2} \quad (30)$$

Orbital trajectory and Keplerian parameters

The solution of the linear differential equation is a constant plus a sinusoidal term. Choosing the angular origin at periapsis gives the conic equation in its conventional polar form.

$$u(\theta) = \frac{\mu}{h^2}(1 + e\cos\theta) \quad (31)$$

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \quad p = \frac{h^2}{\mu} \quad (32)$$

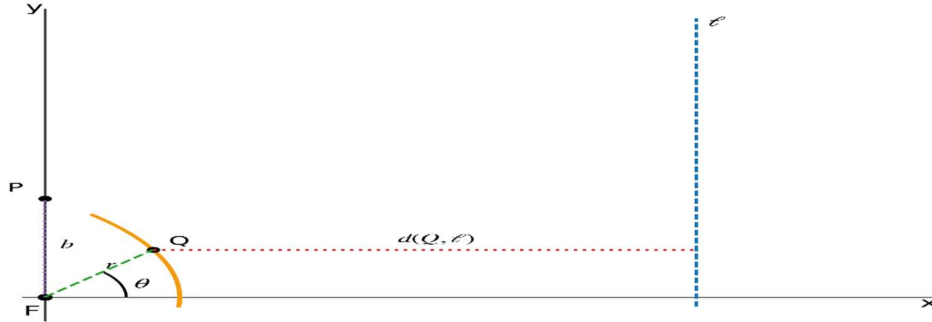


Figure 4: Polar representation of a conic trajectory showing the focal point F, orbital position Q, radial distance r, polar angle θ , and directrix ℓ

Figure 4 presents a conic trajectory using polar coordinates, with the attracting centre at focus F. A point Q on the trajectory is located by its radial distance r and polar angle θ , while the directrix ℓ provides the geometric reference for eccentricity. The orbit is represented by $r(\theta) = p/(1 + e \cos \theta)$, where $p = h^2/\mu$ is the semi-latus rectum. Eccentricity determines the orbit type: $e = 0$ describes a circle, $0 < e < 1$ an ellipse, $e = 1$ a parabola, and $e > 1$ a hyperbola. This geometric construction illustrates the solution of Binet's equation.

For a bound elliptical orbit, the semi-major axis a, semi-minor axis b, and eccentricity e satisfy $b = a\sqrt{1 - e^2}$. The distances at periapsis and apoapsis are $a(1 - e)$ and $a(1 + e)$, respectively.

$$b = a\sqrt{1 - e^2} \quad (33)$$

$$r_p = a(1 - e), \quad r_a = a(1 + e) \quad (34)$$

The specific orbital energy is $\varepsilon = E/m_r$. For a Keplerian conic, $\varepsilon = -\mu/(2a)$ for a bound ellipse. Combining this energy relation with the mean motion gives the orbital period and Kepler's third law.

$$\varepsilon = -\frac{\mu}{2a} \quad (35)$$

$$n = \sqrt{\frac{\mu}{a^3}} \quad (36)$$

$$= \frac{2\pi}{n} = 2\pi\sqrt{\frac{a^3}{\mu}} \quad (37)$$

RESULTS AND DISCUSSION

Effective potential and circular-orbit stability

The Newtonian effective potential contains a positive centrifugal term proportional to r^{-2} and a negative gravitational term proportional to r^{-1} . Their balance produces a stationary radius $r_c = h^2/\mu$. The second derivative at this radius is positive, so the stationary point is a local minimum. Small radial perturbations at fixed angular momentum are therefore stable in the energy description.

$$V_e(r) = \frac{m_r h^2}{2r^2} - \frac{m_r \mu}{r} \quad (38)$$

$$r_c = \frac{h^2}{\mu}, \quad \left. \frac{d^2 V_e}{dr^2} \right|_{r=r_c} > 0 \quad (39)$$

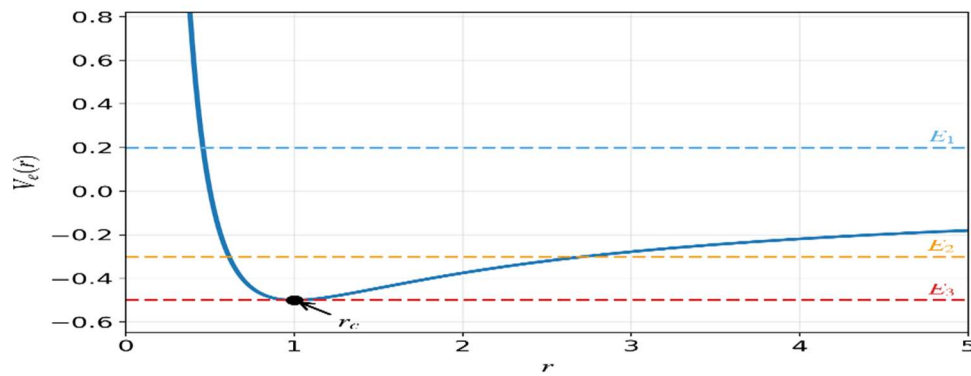


Figure 3. Variation of the effective potential energy $V_e(r)$ with radial distance r for a Newtonian gravitational system, showing the stable circular equilibrium radius r_c and selected mechanical energy levels

Figure 3 shows the effective potential for normalized parameters $L = 1$, $m_r = 1$, and $K = 1$. The curve reaches its minimum at $r_c = 1$, where $V_e(r_c) = -0.5$, before approaching zero from below as radial distance increases. The equilibrium satisfies $dV_e/dr = 0$ and $d^2V_e/dr^2 > 0$, indicating stability against small radial perturbations at fixed angular momentum. The level $E_3 = -0.5$ corresponds to the circular equilibrium, $E_2 = -0.3$ permits radial motion between two turning points, and $E_1 = 0.2$ admits unbounded motion. These are illustrative energy levels rather than measured values.

Orbital trajectories and eccentricity

The trajectory equation is a conic with the attracting body at a focus. Its type is determined by eccentricity and specific orbital energy. A circular trajectory has $e = 0$. A bound ellipse has $0 < e < 1$ and $\varepsilon < 0$. The parabolic limiting trajectory has $e = 1$ and $\varepsilon = 0$. A hyperbolic trajectory has $e > 1$ and $\varepsilon > 0$.

$$e = 0: \text{circular orbit} \quad (40)$$

$$0 < e < 1, \varepsilon < 0: \text{elliptical orbit} \quad (41)$$

$$e = 1, \varepsilon = 0: \text{parabolic trajectory} \quad (42)$$

$$e > 1, \varepsilon > 0: \text{hyperbolic trajectory} \quad (43)$$

For an ellipse, the orbital period depends on the semi-major axis and the standard gravitational parameter. The period does not depend directly on eccentricity for a fixed semi-major axis and fixed central gravitational parameter.

Kepler's laws and planetary periods

Conservation of angular momentum establishes the equal-areas law through the constant areal velocity. For a bound ellipse, the area is πab and the orbital period follows from the areal velocity $h/2$. Combining the area relation with $h^2 = \mu a(1 - e^2)$ gives the period relation.

$$T = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h} \quad (44)$$

$$h^2 = \mu a(1 - e^2), \quad b^2 = a^2(1 - e^2) \quad (45)$$

$$T^2 = \frac{4\pi^2 a^3}{\mu} \quad (46)$$

Table 3 is the ratio of period squared to semi-major axis cubed is normalized to unity.

Table 3: Data illustrating Kepler's Third Law for the planets of the Solar System

Celestial body	a (AU)	T (years)	T ² (years ²)	a ³ (AU ³)	T ² /a ³
Mercury	0.387	0.241	0.058	0.058	1.000
Venus	0.723	0.615	0.378	0.378	1.000
Earth	1.000	1.000	1.000	1.000	1.000
Mars	1.524	1.881	3.540	3.540	1.000
Jupiter	5.203	11.862	140.9	140.9	1.000
Saturn	9.537	29.457	867.0	867.0	1.000
Uranus	19.191	84.020	7059.4	7059.4	1.000
Neptune	30.068	164.8	27183	27183	1.000

The table presents the proportional relationship between orbital period squared and semi-major axis cubed for the normalized values. The theoretical constant in dimensional form is $4\pi^2/\mu$, which depends on the central gravitational parameter and the chosen units.

Discussion

The reduction of the two-body system to relative motion provides a direct connection between gravitational interaction and orbital behaviour. By separating centre of mass translation from relative motion, the Lagrangian formulation expresses the dynamics in terms of the

reduced mass and gravitational parameter. The angular coordinate is cyclic, giving conservation of angular momentum, while the conservative gravitational force ensures constant mechanical energy. These two conserved quantities are complementary: angular momentum governs the angular motion and swept area, whereas the energy integral determines the allowed radial motion. Their combined use offers a consistent interpretation of the orbital trajectory without treating radial and angular behaviour as unrelated processes.

The effective potential makes the stability of circular motion particularly clear. The balance between the angular momentum contribution and gravitational attraction determines the circular radius, $r_c = h^2/\mu$. At that radius, the effective potential has a stationary minimum and positive second radial derivative. This establishes local radial stability for small disturbances at fixed angular momentum. Figure 3 illustrates the associated energy structure using normalized parameters, making visible the distinction between circular equilibrium, bounded radial excursions and unbounded motion. The stability criterion is therefore linked to the curvature of the effective potential rather than to the appearance of a circular path alone.

Binet's equation provides a geometric counterpart to the energy formulation. Expressing radial separation through the orbital angle yields a conic solution for Newtonian inverse-square attraction. The specific angular momentum sets the semi-latus rectum, while eccentricity determines the shape of the trajectory. The associated specific orbital energy separates bounded elliptical motion from parabolic and hyperbolic cases. These relationships show that orbit geometry and energy classification arise from the same governing dynamics. Figure 4 clarifies the polar construction and the role of the focus, radial distance and orbital angle in representing the resulting trajectory.

The temporal description follows naturally from conservation of angular momentum. Constant areal velocity explains the change in orbital speed along an elliptical path and establishes Kepler's second law. For a bound ellipse, the combination of areal velocity and ellipse geometry gives $T^2 = 4\pi^2 a^3/\mu$, demonstrating how the orbital period depends on the semi-major axis and gravitational parameter. Table 3 illustrates the corresponding proportionality through planetary periods and orbital dimensions. Taken together, the energy, trajectory and period relations present a unified analytical account of orbital motion that remains useful for interpreting planetary and satellite dynamics under the defined Newtonian assumptions.

CONCLUSION

The Newtonian two-body analysis establishes consistent relationships among gravitational force, conserved quantities, orbital stability and trajectory geometry. Separation into centre of mass and relative coordinates simplifies the governing motion, while the Lagrangian framework leads to conservation of angular momentum and mechanical energy. The effective potential identifies a stable circular equilibrium for small radial disturbances at fixed angular momentum. Through Binet's equation, the same dynamics yields conic trajectories characterized by eccentricity and specific orbital energy. Conservation of areal velocity also provides the basis for Kepler's second law and the period relation of the third law. The combined findings show that orbital stability, shape and timing can be understood through a common analytical description. This formulation offers a clear foundation for interpreting ideal gravitational trajectories in celestial mechanics and for developing more specialized analyses of orbital motion.

- i. Orbital perturbations. Extend the analytical baseline to include selected third-body attractions and nonspherical gravity terms. Compare changes in orbital elements and local stability conditions with those of the unperturbed Keplerian solution.
- ii. Numerical verification. Integrate the relative equations of motion for clearly specified initial conditions spanning circular, elliptical and escape trajectories. Assess agreement with the analytical conics, conservation laws and turning-point predictions using explicit numerical error measures.
- iii. Extended physical models. Examine dissipative or tidal forces where they are relevant to the chosen orbital system, with relativistic corrections reserved for regimes that require them. Establish which classical invariants cease to hold and how this affects orbital interpretation.

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