



Finite Models of Two-Dimensional Topological Spaces Using Directed Graphs

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ABSTRACT

This article develops a finite directed-graph framework for representing selected topological spaces through an associated abstract simplicial complex. We distinguish arbitrary reflexive relations from preorders: reflexivity alone does not define a finite topology, and antisymmetry and transitivity are not assumed unless explicitly stated. For a finite looped digraph D , we define $K(D)$ by directed chains and verify that the resulting collection is closed under taking faces. Using this construction, we describe graph models of the circle, cylinder, torus, cone, and sphere, and check their simplex counts and topological invariants. We also formulate a dominated-vertex reduction criterion directly in terms of $K(D)$; every permitted deletion preserves the homotopy type of its geometric realization, and the finite reduction algorithm terminates. We also give the standard spanning-tree presentation for $\pi_1(|K(D)|)$. The displayed constructions are not claimed to be minimal unless a separate lower-bound argument is provided. The principal computations yield $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^2) = 1$, and $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$.

INTRODUCTION

Finite topological spaces provide a combinatorial setting for algebraic topology. The classical correspondence between finite T_0 spaces and partially ordered sets connects finite-space topology with order complexes and simplicial homotopy theory (McCord, 1966; Stong, 1966). Barmak (2011a) develops the algebraic topology of finite spaces, while Barmak and Minian (2007, 2008) study minimal finite models and reductions that preserve simple homotopy type. Barmak (2011b) also studies Quillen's Theorem A in the poset setting. These results are formulated for finite topological spaces or posets, where the underlying relation has order-theoretic properties.

A directed graph permits a more general input: its edge relation may be reflexive without being transitive or antisymmetric. This flexibility is useful for encoding directional adjacency, but a reflexive relation by itself does

not define a finite topology. A topological interpretation therefore requires an explicit construction of an associated simplicial complex. Related work includes graph homotopy for undirected graphs (Babson et al., 2006), relation-based methods for posets (Fernández & Minian, 2020), and digital graph models of manifolds (Evako, 2017); these frameworks are related but are not identical to the directed-chain construction used here. The research gap addressed in this paper is to specify that construction rigorously, verify selected graph realizations, and give a reduction criterion whose effect on the associated complex is proved directly. We do not claim that the displayed graphs are universally smaller than established finite topological models.

This study aims to develop and examine finite directed-graph models of selected topological spaces, investigate generalized beat-point reductions, and establish

procedures for determining their associated fundamental groups.

RESEARCH METHODOLOGY

This is a theoretical, combinatorial study. The method has four stages: (i) specify a finite looped digraph D ; (ii) construct its associated abstract simplicial complex $K(D)$ and verify closure under faces; (iii) identify the geometric realization using the stated product, cone, or suspension construction and check simplex counts and topological invariants; and (iv) perform only those vertex deletions that satisfy the dominated-vertex criterion and compute fundamental groups using a spanning tree and the 2-simplex relations. All topological claims below concern $|K(D)|$. The edge relation is treated as a finite topological-space order only when the additional preorder conditions are stated.

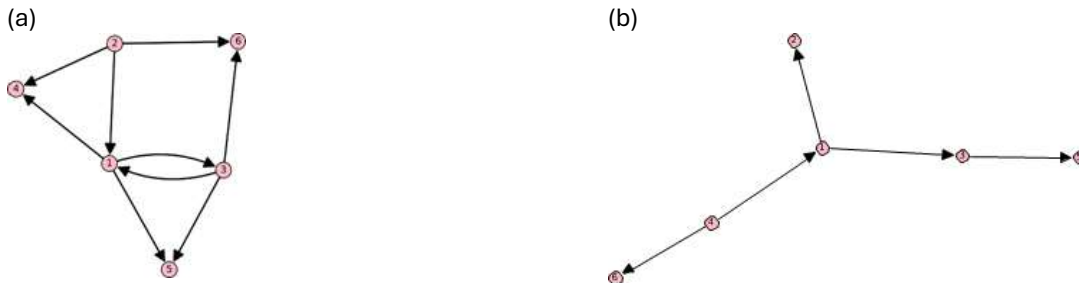


Figure 1: Examples of directed graphs (a) and (b).

Definition 2.4 (Subgraph). A digraph $D_1 = (V_1, E_1)$ is a subgraph of $D = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Definition 2.5 (Induced subgraph). For $V_1 \subseteq V$, the induced subgraph $D[V_1]$ has vertex set V_1 and edge set $E \cap (V_1 \times V_1)$.

Directed Graphs and Relations

Definition 2.1 (Reflexive digraph). A finite reflexive digraph is a pair $D = (V, E)$, where V is a finite nonempty set and $E \subseteq V \times V$ contains every loop (v, v) , $v \in V$. Equivalently, E is a reflexive binary relation on V .

We write $x \rightarrow y$ when $(x, y) \in E$ and call this an edge from x to y . Unless explicitly stated otherwise, no transitivity or antisymmetry assumption is imposed.

Definition 2.2 (Homomorphism). Let $D = (V, E)$ and $D' = (V', E')$ be digraphs. A function $f: V \rightarrow V'$ is a digraph homomorphism if $x \rightarrow y$ in D implies $f(x) \rightarrow f(y)$ in D' .

Definition 2.3 (Directed graph). A directed graph is a pair $D = (V, E)$, where V is the vertex set and $E \subseteq V \times V$ is the edge set. The order of an edge matters: (x, y) represents an arrow $x \rightarrow y$. In this article, the graphs used to construct $K(D)$ are reflexive, so all loops are present even when they are omitted from a drawing.

Example 1. The following figures show two directed graphs.

Thus every edge of D whose endpoints both lie in V_1 is retained.

Example 2. The subgraph generated by the vertices $\{1, 3, 5, 4\}$ in Figure 1(a) is the induced subgraph $D[\{1, 3, 5, 4\}]$.

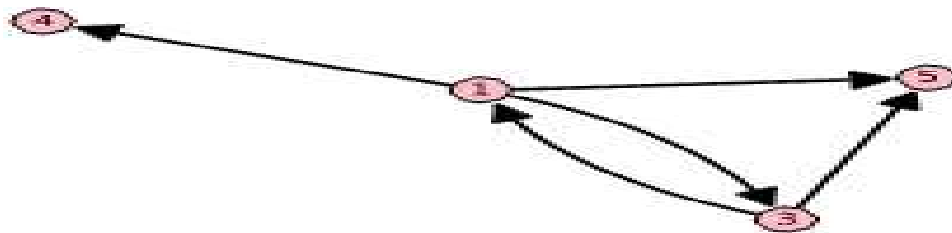


Figure 2: Induced subgraph $D[\{1, 3, 5, 4\}]$

Definition 2.6 (Graphical representation of a relation). A relation on X is represented by a digraph whose vertices are the elements of X and whose arrows are exactly the pairs (x, y) in the relation.

Observations

In drawings of reflexive digraphs, loops may be shown or omitted for readability; mathematically, every vertex has a loop.

A general digraph can be made reflexive by adding all loops. This convention does not add transitivity or antisymmetry, and the terms “digraph” and “reflexive digraph” are used interchangeably below only when the loop convention is in force.

Example 3. Let $F = \{0, 1, 2, \dots, 9\}$. Include every loop and orient each edge of the 10-cycle from its even-labelled endpoint to its adjacent odd-labelled endpoint.

Precisely, $x \rightarrow y$ if $x = y$, or if x is even, y is odd, and x and y are adjacent in the cycle $0-1-2-\dots-9-0$.

The resulting directed graph is shown in Figure 3.

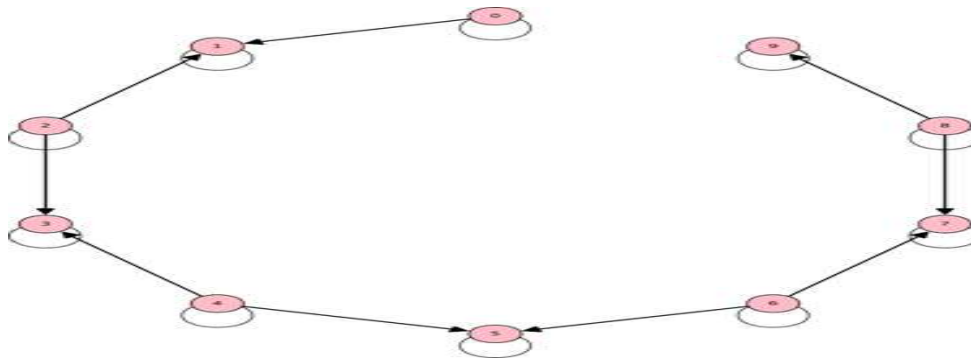


Figure 3: Directed graph associated with F

The Associated Simplicial Complex and Geometric Realization

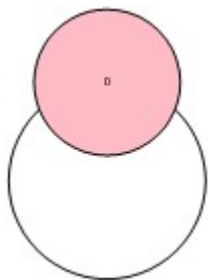
We associate a simplicial complex to each finite reflexive digraph. This construction is defined directly from directed chains and does not require the edge relation to be transitive.

A nonempty set of vertices is a directed chain when its vertices can be ordered so that every earlier vertex points to every later vertex. This definition is used to construct an abstract simplicial complex, rather than to assert that an arbitrary digraph is itself a finite topological space.

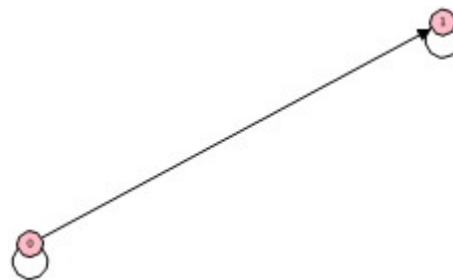
Definition 2.7 (Directed simplex). Let $D = (V, E)$. A nonempty subset $\sigma = \{v_0 \dots v_k\} \subseteq V$ is a k -simplex if its vertices admit an ordering $v_0 \dots v_k$ such that $v_i \rightarrow v_j$ for every $i < j$.

Definition 2.8 (Associated simplicial complex). Let $K(D)$ be the collection of all directed simplices of D , together with the empty simplex. Every subset of a directed simplex inherits the restricted ordering, so $K(D)$ is closed under taking faces and is an abstract simplicial complex. If the relation is a partial order, $K(D)$ is the usual order complex; for a general digraph, we use only the definition above. The geometric realization is denoted by $|K(D)|$. (1) $K(D) = \{\emptyset\} \cup \{\sigma \subseteq V : \sigma \neq \emptyset \text{ and } \sigma \text{ admits an ordering } v_0, \dots, v_k \text{ with } v_i \rightarrow v_j \text{ for all } i < j\}$.

The drawings in Figures 4 and 5 illustrate simplices of dimensions 0 through 4 in this construction.

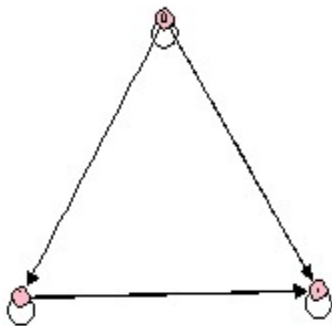


(a)

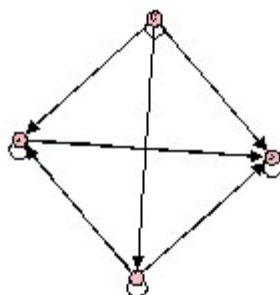


(b)

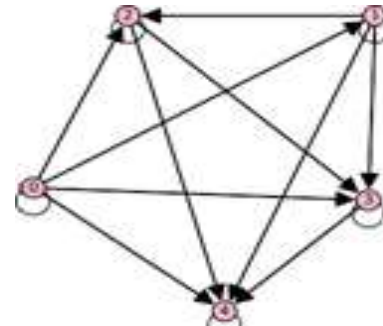
Figure 4: A 0-simplex (a) and a 1-simplex (b).



(a)



(b)



(c)

Figure 5: A 2-simplex (a), a 3-simplex (b), and a 4-simplex (c).

Example 4. For the digraph A in Figure 6(a), the associated simplicial complex $K(A)$ and its geometric realization $|K(A)|$ are represented in Figure 6(b).

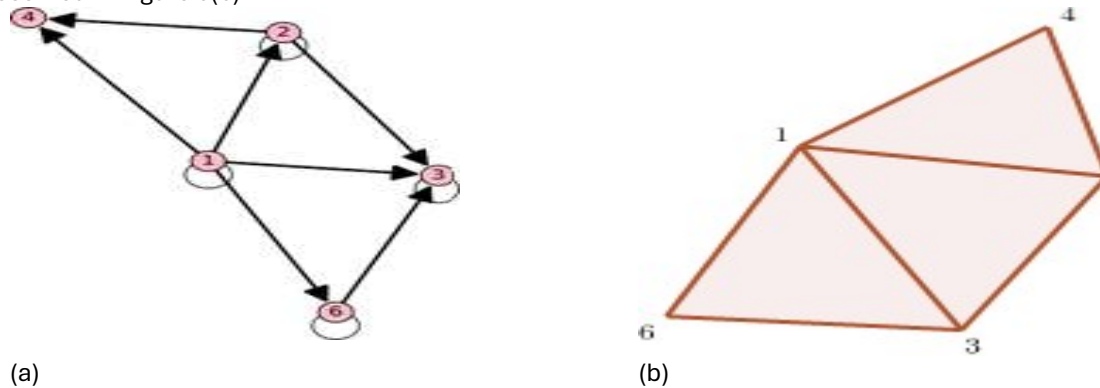


Figure 6. Directed graph A (a) and geometric realization $|K(A)|$ (b).

Definition 2.9 (Finite graph model). A finite digraph D is a graph model of a topological space X if $|K(D)|$ is homeomorphic to X . If only homotopy equivalence is established, we call D a homotopy model of X . Homeomorphism and homotopy equivalence are distinct and are not used interchangeably. A model is called minimal only after proving that no model of the same specified type uses fewer vertices; no general minimality claim is made for the examples below.

The constructions below use directed cycles, coordinatewise products, cones, and suspensions. Their

topological types are verified from these constructions and the combinatorial checks stated after Figure 11.

We now give explicit graph models for the circle, cylinder, torus, cone, and sphere.

Graph Constructions of Selected Spaces

Example 5 (Circle). Let C_3 have vertices $\{0, 1, 2\}$, all loops, and directed edges $0 \rightarrow 1$, $1 \rightarrow 2$, and $2 \rightarrow 0$. Its associated complex has three vertices and three edges but no 2-simplex, so $|K(C_3)|$ is a triangulated circle, homeomorphic to S^1 .

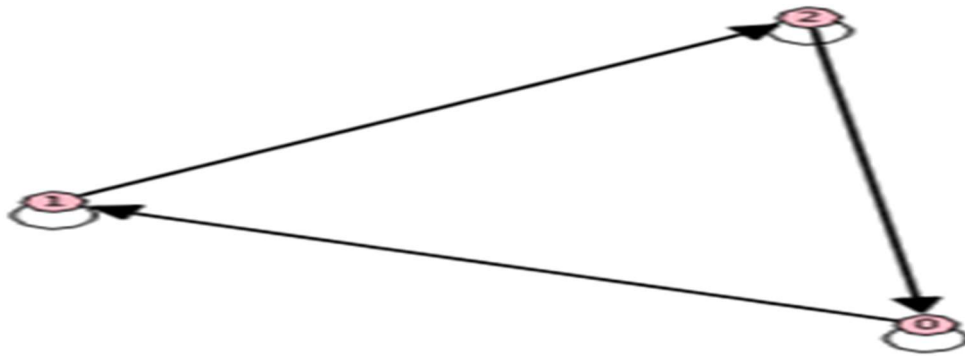
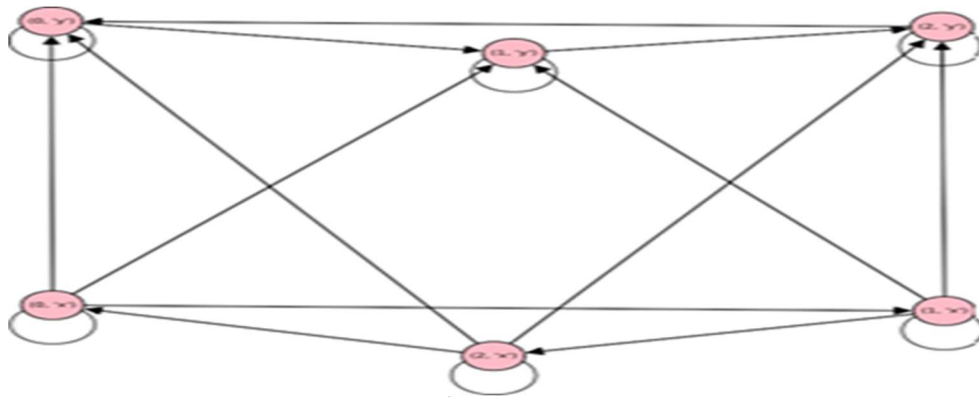


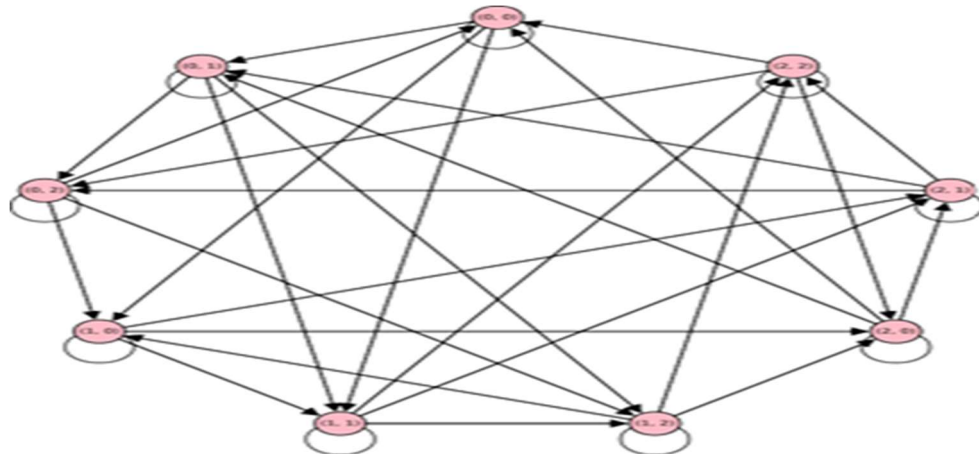
Figure 7: Three-vertex graph model of S^1

Definition 2.10 (Cartesian product). Let $D = (V, E)$ and $D' = (V', E')$ be reflexive digraphs. The product $D \times D'$ has vertex set $V \times V'$ and the coordinatewise relation $(x, y) \rightarrow (x', y')$ if and only if $x \rightarrow x'$ in D and $y \rightarrow y'$ in D' .

Example 6 (Cylinder). Take the product of the directed 3-cycle C_3 with the two-vertex interval digraph $I = \{0, 1\}$, with $0 \rightarrow 1$ and all loops. The associated complex gives a triangulation of $S^1 \times I$, the cylinder, as shown in Figure 8.

Figure 8: Graph model of the cylinder $S^1 \times I$

Example 7 (Torus). The coordinatewise product $C_3 \times C_3$ gives the graph model shown in Figure 9. Its associated complex is the product triangulation of $S^1 \times S^1$.

Figure 9: Graph model of the torus $S^1 \times S^1$

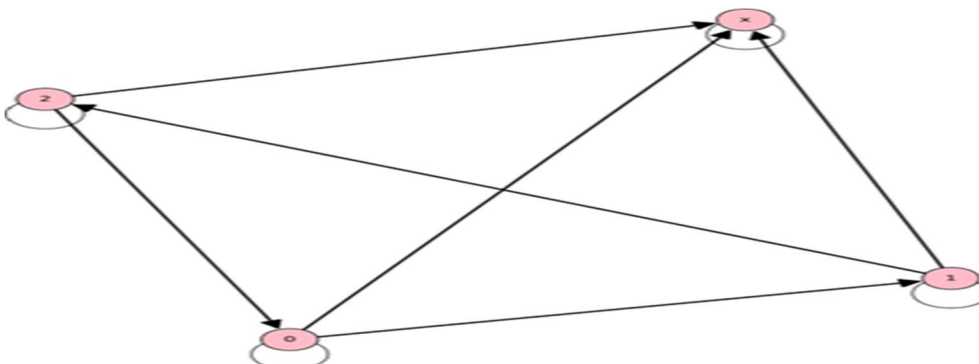
We next define graph operations whose associated complexes are the cone and suspension of the original complex.

Definition 2.11 (Cone). For $D = (V, E)$, add a new vertex a and all edges $v \rightarrow a$ for $v \in V$, retaining the original edges and all loops. Denote the resulting digraph by $C(D)$.

Every simplex of $K(D)$ extends to a simplex with a , so $K(C(D))$ is the simplicial cone on $K(D)$.

Observation. Graphically, the cone operation adds one new vertex and an arrow from each original vertex to that vertex.

Example 8 (Cone). Applying this construction to C_3 gives the graph model of $C(S^1)$ shown in Figure 10; its realization is a disk and is contractible.

Figure 10: Graph model of the cone $C(S^1)$

Definition 2.12 (Suspension). For $D = (V, E)$, add two new vertices p and q , retain all original edges and loops, and add $v \rightarrow p$ and $v \rightarrow q$ for every $v \in V$. No edge is added between p and q . Denote the resulting digraph by $S(D)$.

The associated complex $K(S(D))$ is the union of two cones on $K(D)$, glued along $K(D)$; hence its realization is the suspension of $|K(D)|$.

Example 9 (Sphere). The suspension of C_3 gives the five-vertex graph model of S^2 shown in Figure 11.

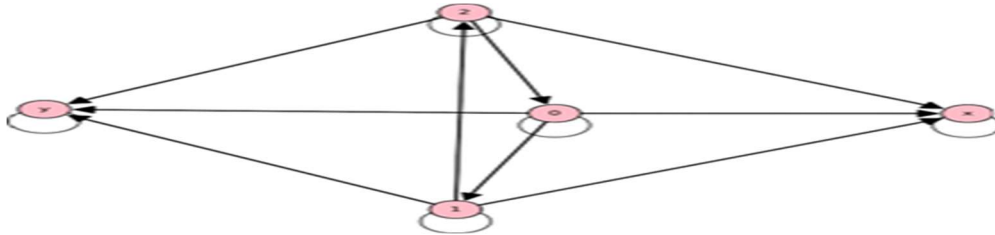


Figure 11: Graph model of S^2 as the suspension of S^1

Combinatorial verification of the displayed models. The following counts refer to nondegenerate simplices of $K(D)$, written as (f_0, f_1, f_2) ; the Euler characteristic is $\chi = f_0 - f_1 + f_2$.

Circle, Figure 7: $(3, 3, 0)$, $\chi = 0$, $\beta_0 = \beta_1 = 1$, and $\pi_1 \cong \mathbb{Z}$.

Cylinder, Figure 8: $(6, 12, 6)$, $\chi = 0$, with six boundary edges. The three rectangles in $S^1 \times I$ are each split into two triangles, giving a triangulation of the cylinder; hence $\pi_1 \cong \mathbb{Z}$.

Torus, Figure 9: $(9, 27, 18)$, $\chi = 0$. The periodic 3×3 product grid has nine horizontal edges, nine vertical edges, nine diagonals, and two triangles in each of its nine squares. Every edge lies in two triangles and each vertex link is a cycle. With the product identifications this is a connected closed orientable surface of Euler characteristic 0, hence a torus; $\beta_1 = 2$ and $\pi_1 \cong \mathbb{Z}^2$.

Cone, Figure 10: $(4, 6, 3)$, $\chi = 1$; the realization is contractible.

Sphere, Figure 11: $(5, 9, 6)$, $\chi = 2$. The complex is a triangular bipyramid, equivalently the suspension of a 3-cycle, so its realization is S^2 and $\pi_1 = 1$. These checks support the stated topological types but do not establish vertex-minimality for every model.

The Klein bottle, Möbius band, and projective plane are not constructed in the present paper. Each would require a specified graph and a proof that the realization of its associated complex has the claimed topology.

Beat-Vertex Reductions and Graph Cores

Beat-point reductions in finite T_0 spaces and posets are supported by order-theoretic hypotheses (Barmak, 2011a; Barmak & Minian, 2008; Stong, 1966). An arbitrary reflexive digraph need not be a poset, so those theorems cannot be transferred without proof. We instead formulate a reduction criterion directly for the associated simplicial complex $K(D)$.

The key idea is to delete a vertex only when it is dominated in $K(D)$. This ensures that the deletion preserves the

homotopy type of the geometric realization, rather than relying on an unverified local edge condition.

The reduction algorithm below is finite and reproducible from the stated criterion. The terminal digraph may depend on the sequence of valid deletions; no uniqueness claim for the reduced digraph is used.

Dominated Vertices and the Reduction Criterion

Definition 3.1 (Simplicial beat vertex). Let D be a finite reflexive digraph and let x, y be distinct vertices. We say that x is dominated by y (or is a simplicial beat vertex with dominator y) if, for every simplex $\sigma \in K(D)$ with $x \in \sigma$ and $y \notin \sigma$, the set $\sigma \cup \{y\}$ is also a simplex of $K(D)$.

Theorem 3.1 (Homotopy-preserving deletion). If x is dominated by y in $K(D)$, then deleting x and all incident edges gives an induced subgraph $D[V \setminus \{x\}]$ whose associated complex is homotopy equivalent to $K(D)$.

Proof. Define $r: K(D) \rightarrow K(D[V \setminus \{x\}])$ on vertices by $r(x) = y$ and $r(v) = v$ for $v \neq x$. The domination condition ensures that the image of every simplex is a simplex, so r is simplicial. The inclusion i fixes every remaining vertex and $r \circ i$ is the identity. Moreover, the identity map and $i \circ r$ are contiguous because each simplex together with its image under $i \circ r$ is a simplex. Their geometric realizations are therefore homotopic, and the deletion preserves homotopy type.

Definition 3.2 (Reduction core). A reduction core of D is any induced subgraph obtained by a finite sequence of deletions satisfying Definition 3.1 and containing no further vertex that is dominated under that criterion.

Each deletion preserves the homotopy type of $|K(D)|$. This definition does not assert that the terminal digraph is unique up to isomorphism or homeomorphism.

Algorithm 3.1 (Dominated-vertex reduction).

Input: a finite reflexive digraph $D = (V, E)$.

Step 1. Construct $K(D)$ by checking each nonempty subset $\sigma \subseteq V$ and each ordering of its vertices; σ is a simplex exactly when every earlier vertex points to every later vertex.

Step 2. Search for distinct vertices x and y such that $\sigma \cup \{y\} \in K(D)$ for every simplex σ containing x but not y .

Step 3. If such a pair exists, delete x and all edges incident to x , record the pair (x, y) , recompute the induced complex, and return to Step 2.

Step 4. If no such pair exists, stop and output the terminal induced subgraph and the recorded deletion sequence.

Termination. Each iteration removes one vertex, so the procedure terminates after at most $|V| - 1$ deletions if a nonempty graph is retained. The domination test and the

recomputation of $K(D)$ are finite, although exhaustive enumeration may be expensive for large graphs.

The procedure is software-independent. A reproducible implementation stores each simplex as a finite subset of V and tests the defining inclusion condition for every ordered pair (x, y) ; because V is finite, exhaustive enumeration suffices, although it may be costly for large graphs.

The following examples illustrate the reduction criterion.

Example 10. For the digraph A in Figure 6(a), the sequence in Figure 12 deletes vertices 2, 3, 4, and 1 in turn, leaving the single vertex 6. Each deletion is a dominated-vertex deletion, so $|K(A)|$ is contractible.

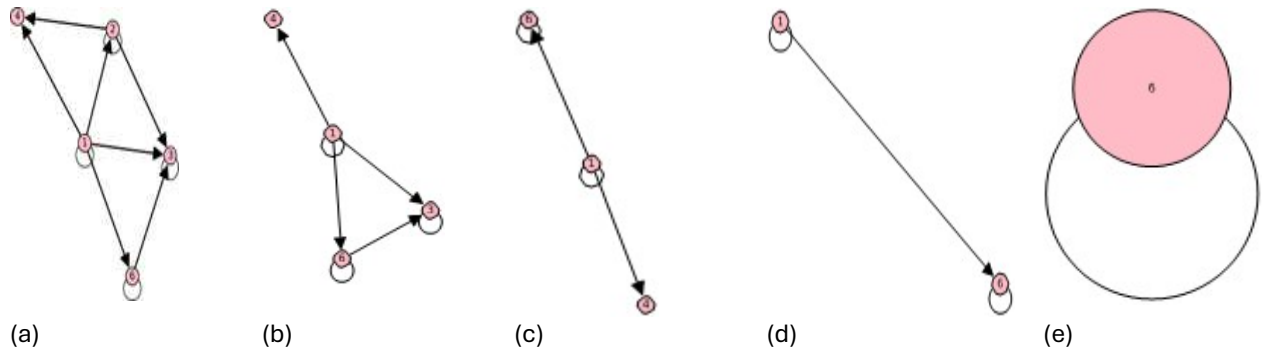


Figure 12: Dominated-vertex reductions of A : (a) A , (b) $A \setminus \{2\}$, (c) $A \setminus \{2, 3\}$, (d) $A \setminus \{2, 3, 4\}$, and (e) $A \setminus \{2, 3, 4, 1\}$.

Definition 3.3 (Contractible digraph). A finite digraph D is called contractible when $|K(D)|$ is a contractible topological space. A reduction to a single vertex is a certificate of contractibility.

Proposition 3.2 (Cone contractibility). For every finite reflexive digraph D , the cone $C(D)$ is contractible.

Proof. By Definition 2.11, $K(C(D))$ is a simplicial cone with apex a . A simplicial cone is contractible, so $|K(C(D))|$ is contractible.

Example 11. Figure 13 gives a digraph G , and Figure 14 records a vertex-deletion sequence for it. The simplex-extension checks for each deletion are not listed, so the sequence is retained as a graphical illustration only and is not used to support a homotopy-equivalence claim.

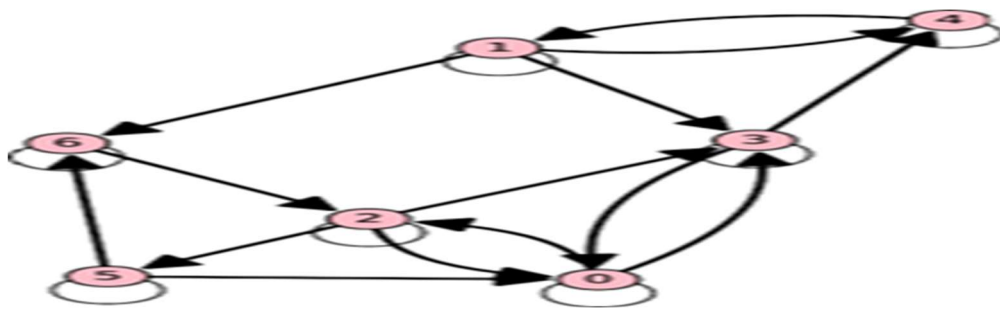
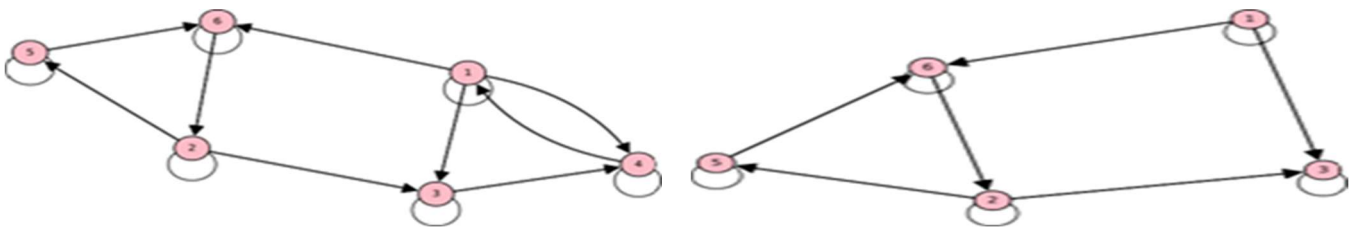


Figure 13: Directed graph G



(a)

(b)

Figure 14: A vertex-deletion sequence displayed for G : (a) $G \setminus \{0\}$; (b) the resulting subgraph

A claim that two graph models are not homotopy equivalent must be supported by a comparison of an invariant of their associated realizations, such as the fundamental group or homology. No uniqueness theorem for terminal digraphs is assumed here.

Fundamental Group of the Associated Simplicial Complex

Throughout this section, $\pi_1(D)$ is shorthand for $\pi_1(|K(D)|)$, the fundamental group of the geometric realization; it is not a separate graph-homotopy invariant. Let T be a spanning tree of the 1-skeleton of $K(D)$, and orient each 1-simplex. Tree edges represent the identity. Each edge e outside T gives a generator g_e .

Proposition 4.1 (Spanning-tree presentation). The group $\pi_1(|K(D)|)$ has one generator g_e for each 1-simplex e not in T and one relation for each 2-simplex.

If a 2-simplex has vertices ordered u, v, w with edges $u \rightarrow v$, $v \rightarrow w$, and $u \rightarrow w$, its boundary gives the relation $g_{uv} g_{vw} g_{uw}^{-1} = 1$, or equivalently $g_{uv} g_{vw} = g_{uw}$. Any edge in T is assigned the identity.

Thus $\pi_1(|K(D)|) \cong \langle g_e : e \notin T \mid g_{uv} g_{vw} = g_{uw} \text{ for every 2-simplex } \{u, v, w\} \rangle$. (2)

Proof. Collapsing the spanning tree identifies the 1-skeleton with a wedge of circles, one for each non-tree edge. The attaching loop of each 2-simplex imposes the stated boundary relation. Cells of dimension greater than two do not change the fundamental group. This is the standard presentation for the fundamental group of a simplicial complex (Munkres, 2000).

Observations.

- In the drawings for this section, reflexive loops are omitted. A geometric 1-simplex is counted once; traversing it in the opposite direction uses the inverse generator, rather than introducing a second generator.
- For each example, choose a spanning tree of the 1-skeleton, set all tree-edge labels to the identity, and then impose the relations supplied by the 2-simplices.

Example 12 (Circle). For the three-vertex model of S^1 in Figure 7, choose two of the three edges as a spanning tree. The remaining edge gives one generator a , and there are no 2-simplices to impose a relation. Hence $\pi_1(S^1) \cong \langle a \rangle \cong \mathbb{Z}$.

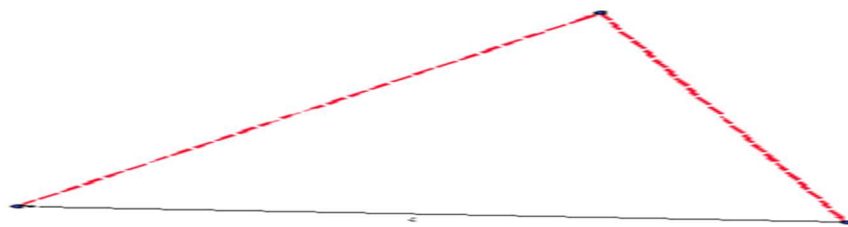


Figure 15: Spanning-tree presentation for the circle model

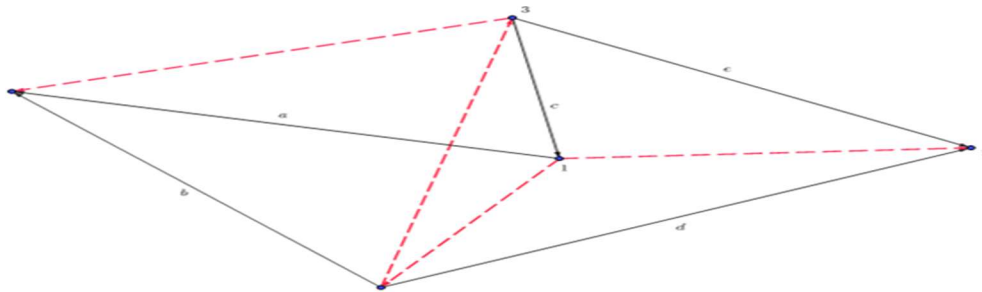


Figure 16: Spanning-tree presentation for the sphere model

Therefore $\pi_1(|K(C_3)|) \cong \mathbb{Z} \cong \pi_1(S^1)$.

Example 13 (Sphere). Let H be the suspension model of S^2 in Figure 11, with equatorial vertices a, b, c in the directed cycle $a \rightarrow b \rightarrow c \rightarrow a$ and suspension vertices p and q . Choose the tree edges ap, bp, cp , and aq .

Denote the non-tree edge generators by $g_{ab}, g_{bc}, g_{ca}, g_{bq},$ and g_{cq} . The six triangular faces give $g_{ab} = g_{bc} = g_{ca} = 1$ from the three faces incident to p , and $g_{bq} = g_{cq} = 1$ from the faces incident to q ; the remaining face relation is then redundant.

All generators are therefore trivial, so $\pi_1(|K(H)|) = 1 \cong \pi_1(S^2)$.

Therefore $\pi_1(G) = \langle \{a, b\} \rangle \cong \mathbb{Z} * \mathbb{Z}$.

Example 14 (Torus). For the product graph in Figure 9, $K(C_3 \times C_3)$ is the stated triangulation of $S^1 \times S^1$. Therefore the product formula for fundamental groups gives $\pi_1(|K(C_3 \times C_3)|) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2$.

Discussion

The constructions in this paper are related to, but distinct from, classical finite-space models. McCord (1966) and Stong (1966) connect finite topological spaces with simplicial complexes, while Barmak (2011a) and Barmak

and Minian (2007, 2008) develop minimal models and reduction methods under the hypotheses appropriate to finite spaces and posets. Here the starting object is a reflexive digraph that need not be transitive or antisymmetric; its topological meaning is supplied explicitly by $K(D)$. This is the principal methodological distinction, not a claim that these graphs are new minimal finite topological spaces. The construction also differs from the intersection-graph approach to digital manifolds studied by Evako (2017), and from the graph-homotopy framework of Babson et al. (2006). The verified results are the displayed realizations of S^1 , $S^1 \times I$, $S^1 \times S^1$, the cone, and S^2 ; the dominated-vertex deletion theorem; and the spanning-tree presentation with the circle and sphere calculations. Limitations remain: the method requires constructing $K(D)$, exhaustive enumeration can be costly, no uniqueness theorem for the terminal digraph is proved, and minimality is not established for all displayed models. The Klein bottle, Möbius band, and projective plane are not explicitly modeled here. Future work can investigate efficient domination tests, conditions that translate the simplicial criterion into local edge conditions, and extensions to higher homotopy groups.

CONCLUSION

This paper defined an associated abstract simplicial complex $K(D)$ for a finite reflexive digraph and used it to construct graph models of the circle, cylinder, torus, cone, and sphere. The simplex counts and topological invariants verify the stated realizations: the circle has $\pi_1 \cong \mathbb{Z}$, the sphere has trivial fundamental group, and the torus has $\pi_1 \cong \mathbb{Z}^2$. A simplicial beat-vertex criterion was given and proved to preserve the homotopy type under each permitted deletion; the resulting reduction algorithm terminates after finitely many steps. The work does not claim minimality for every model or uniqueness of the terminal digraph, and it distinguishes graph-based simplicial models from finite topological spaces whose relations must satisfy additional order-theoretic conditions. Further work will address efficient reductions and higher homotopy invariants.

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