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Distance Transfer and Exact Convex Hop Domination in Powers of Paths

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ABSTRACT
Graph powers change vertex distances in a way that can be described directly from the metric of the original graph. This makes them useful for examining domination parameters defined by distance. Here we study convex hop domination in powers of paths and determine its value for all relevant values of n and k. The analysis begins with the relation $dG^k(u, v) = \lceil dG(u, v)/k \rceil$. Thus, distance two in G^k corresponds in the original graph to $k < dG(u, v) \le 2k$. For path powers, this distance condition can be considered together with geodesic convexity. Arithmetic progressions with step k give the required constructions, while the positions of their endpoints provide the lower bounds. For $k \ge 2$, we obtain $\gamma conh(P_n^k) = n$ for $2 \le n \le k + 1$; $\gamma conh(P_n^k) = \min\{k + 1, 2k - n + 4\}$ for $k + 2 \le n \le 2k$; and $\gamma conh(P_n^k) = \max\{3, \lceil (n - 1)/k \rceil - 3\}$ for $n \ge 2k + 1$. The case $k = 1$ is handled separately and gives $\gamma conh(P_n) = \max\{2, n - 4\}$. These cases determine the convex hop domination number for every nontrivial power of a path. The formulas also give explicit optimal certificates. As a separate computational check, exhaustive enumeration was carried out for 102 instances with $1 \le k \le 6$ and $2 \le n \le 18$; every computed minimum agreed with Theorem 23.

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INTRODUCTION

Domination is a well-established topic in graph theory, with structural, algorithmic, and applied aspects developed extensively in the literature (Haynes et al., 1998). Several distance-based variants replace ordinary adjacency with a prescribed distance condition. In hop domination, a set S is required to reach every vertex outside S at distance exactly two. Ayyaswamy et al. (2015) obtained early bounds for trees, and Henning and Jafari Rad (2017) later studied structural and computational aspects of the parameter, including NP-completeness for restricted graph classes.

Graph convexity supplies the other metric component used in this study. Basic notions of graph distance and intervals are presented in Buckley and Harary (1990). Geodesic convexity was studied by Harary and Nieminen (1981) and is treated more fully in Pelayo's monograph (Pelayo, 2013). Hassan et al. (2023) introduced convex hop domination by requiring a hop dominating set to be geodesically convex. Related work considered weakly convex hop domination (Canoy & Hassan, 2023) and outer-convex hop domination (Isahac et al., 2023). These variants make clear that the convexity condition can substantially restrict the structure of a hop dominating set. Graph powers give a natural setting for examining this interaction. For a positive integer k, the kth power G^k connects two distinct vertices whenever their distance in G is at most k. Structural descriptions of powers of paths and cycles are available in Lin et al. (2011), while chordal powers and related constructions have been investigated in Brandstädt, et al. (1996) and Brandstädt & Le (2009). Parameters studied specifically on path powers include

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acyclic and star coloring (Etawi et al., 2022) and graph pebbling (Alcón & Hurlbert, 2023). Powers of paths and cycles also appear as spanning structures in extremal graph theory (Hng, 2022). In the literature considered for this work, we did not find an exact determination of the convex hop domination number for powers of paths. The question studied here is therefore how the established parameter γconh behaves when the path is replaced by one of its graph powers.

Our approach starts with the change in distance caused by graph powering and then uses the ordered structure of a path. Powering affects two parts of the problem at once: the pairs that occur at hop distance two and the vertices that belong to geodesics in the powered graph. The first can be expressed as a distance condition in the original path, while the second imposes restrictions on powered intervals. Keeping these two effects separate makes it possible to construct feasible sets and derive matching lower bounds.

The study has five main contributions. First, we establish the exact distance relation for graph powers and use it to express hop domination as an annular condition in the original graph.G^k. Second, we show that arithmetic progressions with step k have the geodesic structure needed for convexity in ${P_n}^k$. Third, we obtain the stable-range exact formula by matching an explicit progression construction with endpoint-span lower bounds. Fourth, we close the previously delicate diameter-two transition regime by proving the exact value $\min\{k+1, 2k-n+4\}$, and we separately determine the ordinary-path case $k = 1$. Together these results give a complete classification of γconh($P_n{}^k$) for all $n \geq 2$ and $k \geq 1$. Fifth, we provide an explicit optimal-certificate algorithm and an independent exhaustive audit of 102 small instances.

Preliminaries and Definitions

All graphs considered here are finite, simple, undirected, and connected unless stated otherwise. For a graph G, the vertex and edge sets are $V(G)$ and $E(G)$. The distance $dG(u, v)$ is the length of a shortest u-v path. A $u - v$ path of length $dG(u, v)$ is called a u-v geodesic. The interval $I_G[u,v]$ is the set consisting of u, v, and every vertex lying on some u-v geodesic. For $X \subseteq V(G)$, the subgraph induced by X is denoted $G[X]$.

Definition 1. A set $S \subseteq V(G)$ is a hop dominating set if for every vertex $v \in V(G)\backslash S$ there exists $s \in S$ such that $dG(v, s) = 2$. The minimum cardinality of a hop dominating set is the hop domination number $\gamma h(G)$.

Definition 2. A set $C \subseteq V(G)$ is convex if $IG[x, y] \subseteq C$ for every $x, y \in C$. A set is convex hop dominating if it is both convex and hop dominating. The minimum cardinality of such a set is denoted $\gamma conh(G)$.

Definition 3. For an integer $k \geq 1$, the kth power Gk has vertex set $V(G)$, with distinct vertices u and v adjacent in G^k if and only if $dG(u, v) \leq k$.

The graph-power definition is standard (Etawi et al. 2022; Lin et al., 2011; Brandstädt, et al. 1996; Brandstädt & Le, 2009). Note that $G^1 = G$. If k is at least the diameter of G, then G^k is complete. Since a complete graph has no pair of vertices at distance two, every hop dominating set of a complete graph must equal the whole vertex set (Henning & Jafari Rad, 2017). Consequently $\gamma conh(K_n) = n$.

Remark 4. Every convex hop dominating set is a hop dominating set, and therefore $\gamma h(G) \leq \gamma conh(G)$. This basic inequality can be strict because convexity imposes a global geodesic closure condition in addition to distance-two coverage.

Distance Transfer in Graph Powers

The key simplification is that distances in graph powers can be expressed exactly in terms of distances in the original graph. This standard metric behavior is consistent with metric graph theory (Buckley & Harary, 1990) and the graph-power literature (Lin et al., 2011; Brandstädt, et al. 1996). We state and prove the identity explicitly because both hop domination and convexity are metric notions and the formula is used repeatedly below.

Lemma 5 (distance-transfer lemma). Let G be connected and k≥1. For all $u, v \in V(G)$,

$dG^k(u, v) = \lceil dG(u, v)/k \rceil$.

Proof. Let $d = dG(u, v)$. Take a u-v geodesic in G and divide it into consecutive segments of length at most k. Consecutive segment endpoints are adjacent in G^k, giving a G^k −walk of length $\lceil d/k \rceil$. Hence $dG^k(u, v) \leq \lceil d/k \rceil$. Conversely, every edge of G^k joins vertices whose G-distance is at most k. A G^k-path of length r therefore yields, by the triangle inequality in G, $dG(u,v) \leq rk$. Thus $r \geq \lceil d/k \rceil$, and equality follows. The statement is included as a convenient explicit form of the standard metric relation underlying graph powers (Buckley & Harary, 1990; Lin et al., 2011; Brandstädt, et al. 1996).

Corollary 6. For distinct $u, v \in V(G), dG^k(u, v) = 2$ if and only if $k < dG(u, v) \leq 2k$.

Proof. This is immediate from Lemma 5, because $\lceil dG(u, v)/k \rceil = 2$ precisely when $k < dG(u, v) \leq 2k$.

Proposition 7. A set $S \subseteq V(G)$ is hop dominating in G^k if and only if for every $v \in V(G)\backslash S$ there exists $s \in S$ with $k < dG(v, s) \leq 2k$.

Proof. Apply Corollary 6 to each vertex v outside S. The result converts hop domination in a powered graph into an annular covering condition in the original graph.

Proposition 7 is useful beyond paths. It separates the effect of the power operation into two components. The covering requirement can be checked entirely in the original metric of G, whereas convexity must still be checked with respect to geodesics in G^k. This separation is the methodological principle used throughout the paper.

Corollary 8. If $\text{diam}(G) \leq k$, then $\gamma conh(Gk) = |V(G)|$.

Proof. In this case G^k is complete. No vertex outside a proper subset can be at distance two from the subset, so the only hop dominating set, and hence the only convex hop dominating set, is $V(G)$.

Geometry of Powers of Paths

Let P_n be the path with vertices $v_0, v_1, \ldots, v_{n-1}$ in their natural order. For notational convenience we identify v_i with the integer i. Then $dP_n(i,j) = |i-j|$, and Lemma 5 gives

$dP_n{}^k(i,j) = \lceil |i-j|/k \rceil$.

This formula makes path powers especially tractable. The principal optimal sets will be arithmetic progressions with common difference k. These progressions are not chosen merely for convenience: they are geodesically rigid in the powered metric.

Lemma 9. Let $a \geq 0$ and $r \geq 2$ satisfy $a + (r-1)k \leq n-1$, and put $S = \{a, a+k, \ldots, a+(r-1)k\}$. Then S is convex in Pnk.

Proof. Take $x = a + pk$ and $y = a + qk$ in S with $p < q$. Their distance in $P_n{}^k$ is $q - p$. Let $x = z_0, z_1, \ldots, z_m = y$ be any $x-y$ geodesic in $P_n{}^k$. Then $m = q - p$. Each powered edge $z_i z_{i+1}$ satisfies $dP_n(z_i, z_{i+1}) \leq k$. Hence, by the triangle inequality in the original path,

$(q-p)k = dP_n(x,y) \leq \Sigma_{i=0}{}^{m-1} \quad dP_n(z_i, z_{i+1}) \leq mk = (q-p)k$.Equality therefore holds throughout. In particular, every step has original-path length exactly k, and equality in the path triangle inequality forces all steps to proceed monotonically from x toward y. Consequently $z_i = x + ik = a + (p+i)k$ for $0 \leq i \leq m$. Thus every vertex on every x-y geodesic belongs to S, and $IP_n{}^k[x,y] \subseteq S$. Hence S is convex.

Lemma 10. Assume r≥3 and let $S = \{a, a+k, \ldots, a+(r-1)k\}$. Every vertex lying strictly between two consecutive vertices of S is at distance two in Pnk from some vertex of S.

Proof. Let $v = a + jk + t$ with $1 \leq t \leq k-1$. If $j \leq r-3$, use $s = a + (j+2)k$. Then $dP_n(v,s) = 2k - t$, which lies in $(k, 2k)$. If $j = r-2$, use $s = a + (j-1)k$; then $dP_n(v,s) = k + t \in (k, 2k)$. Corollary 6 gives $dP_n{}^k(v,s) = 2$.

Lemma 11. Let $r \geq 3$ and $S = \{a, a+k, \ldots, a+(r-1)k\}$. If $a \leq 2k$ and $n - 1 - [a + (r-1)k] \leq 2k$, then S is hop dominating in $P_n{}^k$.

Proof. Lemma 10 handles all vertices in internal gaps. Consider a vertex $v < a$ and set $t = a - v$. If $1 \leq t \leq k$, then the second selected vertex $a + k$ is at original distance $k + t \in (k, 2k]$. If $k < t \leq 2k$, the first selected vertex a is itself at distance $t \in (k, 2k]$. The condition $a \leq 2k$ guarantees that every left-tail vertex falls into one of these cases. The right tail is symmetric. Hence every vertex outside S is at distance two in $P_n{}^k$ from S.

Corollary 12. Under the assumptions of Lemma 11, S is a convex hop dominating set of $P_n{}^k$.

Proof. Convexity follows from Lemma 9 and hop domination from Lemma 11.

Exact Convex Hop Domination Number of $P_n{}^k$

We now derive the main result. The upper bound comes from the arithmetic-progression construction. The lower bound is controlled by the span of any convex hop dominating set. Both parts of the argument depend on the annular covering condition together with convexity.

Lemma 13. Let S be a convex hop dominating set of $P_n{}^k$, and let p=min S and q=max S. Then $p \leq 2k$ and $q \geq n - 1 - 2k$.

Proof. If $0 \in S$ then $p = 0$. Otherwise, because S hop dominates 0 in $P_n{}^k$, Proposition 7 gives a selected vertex s with $k < s \leq 2k$, so $p \leq 2k$. The argument at n−1 is symmetric.

Lemma 14. If S is convex in $P_n{}^k$ and $p = \min S$, $q = \max S$, then $|S| \geq \lceil (q-p)/k \rceil + 1$.

Proof. Convexity requires S to contain every vertex of every p-q geodesic in $P_n{}^k$. In particular S contains all vertices of at least one such geodesic. By the distance formula, a shortest p-q path has length $\lceil (q-p)/k \rceil$, and therefore contains $\lceil (q-p)/k \rceil + 1$ vertices.

Lemma 15. If $k \geq 2$ and $n \geq 2k+1$, then every convex hop dominating set of $P_n{}^k$ has at least three vertices.

Proof. Suppose first that $S = \{x, y\}$ with $x < y$. If $y - x > k$, then x and y are nonadjacent in $P_n{}^k$, so every shortest x-y path has an internal vertex; convexity would force that vertex into S, a contradiction. Hence $y - x \leq k$. If $y - x < k$, consider two cases. If $x \geq 1$, take $v = x - 1$. Then $v \notin S$, $dP_n(v,x) = 1 \leq k$, and $dP_n(v,y) = 1 + (y-x) \leq k$. Thus v is adjacent in $P_n{}^k$ to both selected vertices and is at powered distance 1, not 2, from each; hence v is not hop dominated. If $x = 0$, take $v = y + 1$. Since $y - x = y < k$, we have y+1≤k; moreover $n \geq 2k+1$ ensures $v \leq k \leq n-1$, so v exists and v∉S. Now $dP_n(v,y) = 1 \leq k$ and $dP_n(v,x) = y + 1 \leq k$, again leaving v at powered distance 1 from both selected vertices. Finally, if $y - x = k$, any original-path vertex v strictly between x and y exists because $k \geq 2$ and satisfies $dP_n(v,x) < k$ and $dP_n(v,y) < k$; hence it is at powered distance 1 from both x and y and is not hop dominated. Therefore no two-vertex convex hop dominating set exists. A one-vertex set is also impossible: because n≥2k+1, for any selected x at least one neighboring path vertex exists, lies outside S, and is at powered distance 1 from x rather than 2. Thus $|S| \geq 3$.

Theorem 16 (main exact formula). *For integers* $k \geq 2$ and $n \geq 2k+1$,

$\gamma_{conh}(P_n{}^k) = \max\{3, \lceil (n-1)/k \rceil - 3\}$.

Proof. Let S be any convex hop dominating set and $p = \min S, q = \max S$. Lemma 13 gives $q - p \geq n - 1 - 4k$. By Lemma 14, $|S| \geq \lceil (n-1-4k)/k \rceil + 1 = \lceil (n-1)/k \rceil - 3$. Lemma 15 supplies the independent lower bound $|S| \geq 3$. Therefore

$\gamma conh(Pnk) \geq \max\{3, \lceil (n-1)/k \rceil - 3\}$.

For the reverse inequality, set $r = \max\{3, \lceil (n-1)/k \rceil - 3\}$. Choose

$a = \max\{0, n-1-2k-(r-1)k\}$.

The definition of r ensures $0 \le a \le 2k$, and $n-1-[a+(r-1)k] \le 2k$. Also $n \ge 2k+1$ and $r \ge 3$ guarantee that $a+(r-1)k \le n-1$. Thus the progression $S=\{a, a+k, \ldots, a+(r-1)k\}$ is well defined. By Corollary 12 it is convex hop dominating, so $\gamma_{conh}(P_n^k) \le r$. Combining the lower and upper bounds proves the formula.

Corollary 17. If $k \ge 2$ and $2k+1 \le n \le 6k+1$, then $\gamma conh(Pnk)=3$.

Proof. In this range $\lceil (n-1)/k \rceil \le 6$, so the second term in Theorem 16 is at most 3. The lower bound 3 therefore determines the value.

Corollary 18. For fixed $k \ge 2$, $\gamma conh(Pnk) = n/k + O(1)$ as $n \to \infty$.

Proof. Theorem 16 differs from $\lceil (n-1)/k \rceil$ by the constant 3 once the latter exceeds 3.

The formula has a useful geometric interpretation. An optimal set is an arithmetic backbone with spacing exactly k. Convexity favors this spacing because it makes powered geodesics unique along the backbone. Hop domination allows two blocks of width 2k, one near each endpoint, to remain outside the span of the selected set. These two effects account for the subtraction of 3 from the normalized path length.

Completion of the Exact Classification

Theorem 16 covers the stable range $n \ge 2k+1$ for $k \ge 2$. It remains to settle two boundary phenomena: the complete and diameter-two regimes for k≥2, and the unpowered case $k=1$. The following results close both gaps and produce a complete exact classification.

Proposition 19 (complete regime). If $2 \le n \le k+1$, then $\gamma_{conh}(P_n^k)=n$.

Proof. Here $n-1 \le k$, so $P_n^k = K_n$. By Corollary 8 the only hop dominating set is the full vertex set.

Lemma 20. Let $k \ge 2$ and $k+2 \le n \le 2k$. Let S be a convex hop dominating set of P_n^k, with $p=\min S$ and $q=\max S$.

If $q-p \le k$, then every vertex of the interval $\{p, p+1, \ldots, q\}$ belongs to S. If $q-p > k$, then S contains p, q, and every common neighbor of p and q, namely the full interval $\{q-k, q-k+1, \ldots, p+k\}$.

Proof. If $q-p \le k$ and v lies between p and q, then $|v-s| \le k$ for every s∈S. If v were outside S, it would be adjacent in P_n^k to every selected vertex and hence could not be hop-dominated at powered distance two; therefore every such v must be selected whenever S is also hop dominating. For the second statement, $q-p>k$ implies $dP_n^k(p,q)=2$. A vertex z lies on a p-q geodesic of length two exactly when it is adjacent to both endpoints, equivalently $q-k \le z \le p+k$. Convexity therefore forces all these common neighbors into S.

Theorem 21. Let $k \ge 2$ and $k+2 \le n \le 2k$. Then $\gamma conh(P_n^k) = \min\{k+1, 2k-n+4\}$.

Proof. The two constructions giving the upper bound are explicit. First, $S_1=\{0,1,\ldots,k\}$ is a clique and hence convex; every vertex $v > k$ satisfies $k < v \le 2k-1$, so $dP_n^k(0,v)=2$. Thus $|S_1| = k+1$. Second, when $n \ge k+3$, let $B=\{n-k-1, n-k, \ldots, k\}$ and $S_2 = \{0, n-1\} \cup B$. The only nonadjacent extreme pair is 0,n−1, and its common-neighbor interval is exactly B, so S_2 is convex. Every unselected left-side vertex is at original distance in $(k, 2k)$ from n−1, and every unselected right-side vertex is similarly at annular distance from 0. Hence S_2 is convex hop dominating and $|S_2| = 2k-n+4. At\ n=k+2$ the first construction is the smaller one, so the stated minimum is always an upper bound.

For the lower bound, let S be any convex hop dominating set, p=min S, and q=max S. If $q-p \le k$, Lemma 20 gives $\{p, \ldots, q\} \subseteq S. If\ p>0$ and $q<n-1$, the outside vertex $p-1$ can be hop-dominated only if some selected vertex is more than k original-path steps away; since q is farthest to the right, $q-p+1>k$, and therefore $q-p=k. If\ p=0<q<n-1$, the vertex q+1 requires $q+1-p>k$, so q=k; the case $p>0, q=n-1$ is symmetric. If $p=0$ and $q=n-1$, then all n vertices lie in S. In every subcase $|S| \ge k+1$.

If $q-p>k$, convexity and Lemma 20 force the common-neighbor interval $\{q-k, \ldots, p+k\}$ into S in addition to the two endpoints p and q. These sets are disjoint because q−p>k. Hence $|S| \ge [2k-(q-p)+1]+2 = 2k-(q-p)+3 \ge 2k-n+4$, since $q-p \le n-1$. Thus every feasible S has size at least min{k+1,2k−n+4}. Together with the two constructions, equality follows.

Proposition 22. For every $n \ge 2$,
$\gamma conh(P_n) = max\{2, n-4\}$.

Proof. Convex subsets of an ordinary path are exactly vertex intervals. A one-vertex set cannot hop dominate a nontrivial path, so $\gamma conh(P_n) \ge 2$. For $n \ge 7$, let a and b be the first and last vertices of a convex hop dominating interval. If $a \ge 3$, vertex 0 is at distance at least three from every selected vertex and is not hop dominated; hence $a \le 2$. Symmetrically $b \ge n-3$. Therefore $|S| \ge b-a+1 \ge n-4$. Conversely, for $n \ge 7$ the interval $S=\{2,3,\ldots,n-3\}$ has size n−4 and hop dominates the four outside vertices: 0 is at distance two from 2, 1 from 3, n−2 from n−4, and n−1 from n−3. For $2 \le n \le 6$, a two-vertex convex hop dominating set exists: take $V(P_2)$ when $n=2, \{0,1\}\ for\ n=3,4, \{1,2\}\ for\ n=5, and\ \{2,3\}$ for $n=6$. Hence the formula holds for all $n \ge 2$.

Theorem 23. Let $n \ge 2$ and $k \ge 1$. Then

$\gamma conh(P_n^k) = max\{2, n-4\},\ if\ k=1;$

$\gamma conh(P_n^k) = n,\ if\ k \ge 2\ and\ 2 \le n \le k+1;$

$\gamma conh(P_n^k) = min\{k+1, 2k-n+4\},\ if\ k \ge 2\ and\ k+2 \le n \le 2k;$

$\gamma conh(P_n^k) = max\{3, \lceil (n-1)/k \rceil - 3\},$
$if\ k \ge 2\ and\ n \ge 2k+1.$

Proof. The four cases are Proposition 22, Proposition 19, Theorem 21, and Theorem 16, respectively. The regimes are disjoint and cover every pair $n \ge 2, k \ge 1$. Thus the convex hop domination number is determined for every power of every nontrivial path.

Remark 24. The two powered regimes meet continuously at $n = 2k + 1$: the stable formula gives 3, while the transition expression $2k - n + 4$ would also give 3. The change in proof technique occurs because P_n^k has diameter two when $n \leq 2k + 1$, where as longer powers contain geodesics of length at least three.

Computational Verification and Certificate Algorithm

The formulas were obtained analytically, and we used exhaustive computation as a separate check. For each pair (n, k), candidate subsets were examined in increasing order of cardinality. Hop domination was checked from the powered-distance formula, whereas convexity was verified by testing the relevant interval $[x, y]$ for each selected pair. The search did not use structural restrictions taken from the proofs. We tested all 102 instances with $1 \leq k \leq 6\ and\ 2 \leq n \leq 18$; in every case, the computed minimum was the value stated in Theorem 23.

Table 1: Exhaustive check of the exact classification

k	tested n	instances	result
1	2–18	17	All match Theorem 23
2	2–18	17	All match Theorem 23
3	2–18	17	All match Theorem 23
4	2–18	17	All match Theorem 23
5	2–18	17	All match Theorem 23
6	2–18	17	All match Theorem 23
Total		102	No counterexample found

To ensure computational reproducibility, the exhaustive verification procedure was implemented in Python. The implementation is deliberately independent of the closed-form expressions established in Theorem 23. For each pair (n, k), it enumerates candidate vertex subsets in increasing order of cardinality, computes distances in P_n^k, and independently tests both hop domination and geodesic convexity. Consequently, the first feasible subset returned by the program provides the computational value of γconh(P_n^k).

Listing 1. Python implementation of the exhaustive verification procedure

```
from itertools import combinations
from math import ceil

def pdist(i, j, k):
    return ceil(abs(i - j) / k)

def hop_ok(S, n, k):
    for v in range(n):
        if v not in S:
            if not any(pdist(v, s, k) == 2 for s in S):
                return False
    return True

def convex_ok(S, n, k):
    for x in S:
        for y in S:
            dxy = pdist(x, y, k)
            for z in range(n):
                lhs = pdist(x, z, k) + pdist(z, y, k)
                if lhs == dxy and z not in S:
                    return False
    return True

def gamma_conh(n, k):
    V = range(n)
    for r in range(1, n + 1):
        for C in combinations(V, r):
            S = set(C)
            if hop_ok(S, n, k) and convex_ok(S, n, k):
                return r, tuple(sorted(S))

for k in range(1, 7):
    for n in range(2, 19):
        value, S = gamma_conh(n, k)
        print(n, k, value, S)
```

The function power_distance evaluates the powered distance directly from the path metric, so the program does not need to construct or store the complete adjacency matrix of P_n^k. The functions is_hop_dominating and is_convex independently implement the two defining conditions of a convex hop dominating set. Finally, convex_hop_number examines subsets in increasing cardinality; therefore, the first accepted subset is necessarily a minimum convex hop dominating set.

The program was run on all 102 cases with $1 \leq k \leq 6$ and $2 \leq n \leq 18$, as listed in Table 1. Each computed minimum agreed with the corresponding value in Theorem 23. This gives a computational check that is independent of the closed-form classification over the tested range.range. The proofs also provide explicit optimal certificates for each regime. For $k = 1$, the intervals described in.

Proposition 25. For every $n \geq 2$ and $k \geq 1$, an optimal convex hop dominating set of P_n^k can be generated in $O(\gamma conh(P_n^k))$ time and $O(1)$ auxiliary space.

Proof. Apply the regime-specific explicit construction summarized above. Each selected vertex is produced by constant-time arithmetic, and no search or optimization routine is required.

A proposed certificate can be checked without constructing every edge of Pnk. Powered distances can be obtained directly from the original path metric. For an arithmetic-progression certificate, convexity is determined by its residue class modulo k, and hop domination is checked using the annular inequalities in Lemma 11. This gives a direct way to verify a certificate independently of the construction procedure.

Discussion and Extensions

Powers of cycles

The path analysis keeps the distance condition for hop domination separate from the geodesic condition for convexity. This also makes the exact constructions easy to check directly. The 102 enumerated cases agreed with the formulas, but the behavior of other graph families need not follow the path pattern. Cycle powers and powered trees are natural next cases because the same distance relation remains available while their geodesic intervals have a different structure.

Proposition 26. $If\ n \leq 2k + 1, then\ \gamma conh(Cnk) = n.$

Proof. The diameter of Cn is floor$(n/2) \leq k$, so C_n^k is complete. Apply Corollary 8.

The small cycle-power cases do not follow a monotone transition pattern. In particular, the convex hop domination number may decrease when the powered diameter reaches two, while nearby parameter values can require larger convex sets because a cycle can have several geodesics between the same pair of vertices. These computations are not sufficient for an exact cycle formula, so the cycle case is left for a separate analysis rather than inferred from the path classification.

General graph classes

Lemma 5 and Proposition 7 apply to every connected graph. They suggest studying convex hop domination of G^k through two separate pieces of information: annular covering in G and the structure of geodesic intervals in G^k. Trees are a natural next family because geodesics are unique in the original graph, although additional geodesics may appear after powering. Distance-hereditary graphs, block graphs, and chordal graphs may also be approachable because their interval structure is comparatively tractable.

Ordinary hop domination gives a useful point of comparison. Since $\gamma_h(G^k) \leq \gamma_{conh}(G^k)$, any lower bound for hop domination transfers automatically. Conversely, a convexity number or hull-number restriction may force stronger lower bounds. Such comparisons could yield parameter inequalities that are independent of a specific graph family.

Complexity

Hop domination is NP-complete even on restricted graph classes (Henning & Jafari Rad, 2017). The complexity of convex hop domination was identified as a natural open direction in the foundational study (Hassan et al., 2023). Graph powers introduce additional structure but do not automatically make the problem easy. One possible question is whether CONVEX-HOP-DOMINATION remains NP-complete when the input is promised to be a square G^2, or when G is a tree and the powered graph is given implicitly. The complete path classification shows that highly structured powers admit closed forms and optimal certificates.

CONCLUSION

This paper develops a distance-transfer framework for convex hop domination in graph powers and gives a complete exact classification for powers of paths. The identity $dG^k(u,v) = \lceil dG(u,v)/k \rceil$ converts powered distance two into the annulus $k < dG(u,v) \leq 2k$ in the original graph. On path powers, rigid k-step progressions control the stable regime, while a complete common-neighbor description resolves the diameter-two transition regime. Together with the ordinary-path case, these arguments determine γconh(P_n^k) $for\ every\ n \geq 2\ and\ k \geq$1.

The argument uses the power operation in two distinct ways. Hop domination is governed by annular coverage in the original metric, whereas convexity depends on geodesics in the powered graph. This distinction is useful beyond the path case because the distance condition can be retained even when the interval structure changes. The complete and transition regimes were treated separately, an explicit certificate procedure was obtained, and the small-instance computation provided an additional check of the formulas. Cycle powers, powered trees, and the complexity of the parameter remain open directions for further study.

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